Properties of Egalitarian Sequences of Committees: Theory and Experiments

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Abstract. We study the task of electing egalitarian sequences of τ committees given a set of agents with additive utilities for candidates available in each of τ levels. We introduce several rules for electing an egalitarian committee sequence as well as properties for such rules. We settle the computational complexity of finding a winning sequence for our rules and classify them against our properties. We obtain sequential election data from existing election data from the literature. Using this data set, we compare our rules empirically and test them experimentally against some selected properties.

1 Prologue

Preferences can be more fine-grained than a single-committee election can represent. For instance, instead of asking faculty members about their representative in an appointment committee, one can ask which candidate they prefer to be elected for each of several *levels* (roles), e.g., (co-)heads, professors, researchers, students. Consequently, people can nominate candidates differing from their 'overall favorite' candidate, which makes it possible to respect their opinions on a finer scale when selecting a committee for each level. Moreover, such *multilevel, sequential* elections allow us to seek for additional fairness criteria fulfilled by the committee sequence. In this work, we seek for egalitarian committee sequences.

Splitting the question about one's favorite representative in a committee into questions about favorite representatives for multiple roles is just one example for a sequential election. Sequential elections can be used for problems with a temporal component as well, for instance which crop farmers prefer to grow in which month, or which activity each member of a travelling group prefers for which day. Scenarios in which people already think in categories can also be modeled this way, e.g., selecting the favorite song, movie, or game in multiple music, movie, or gaming genres, respectively.

Formally, our model receives the following input:

Definition 1. A cumulative[1] (multilevel, sequential[2]) election $\mathcal{E} = (A, \mathcal{C}, U, \kappa)$ consists of: A set A of n agents, a sequence $\mathcal{C} = (C_1, \dots, C_\tau)$ of candidate subsets from a candidate set C of size m not containing $\emptyset$, a sequence $U = (u_1, \dots, u_\tau)$ of utilities $u_t \colon A \times C_t \to \{0, \dots, z_t\}$ such that for each $a \in A$ we have $\sum_{c \in C_t} u_t(a, c) \leq z_t$ and a sequence $\kappa = (k_1, \dots, k_\tau)$ of nonnegative integers.

A committee sequence $\mathcal{X} = (X_1, \dots, X_\tau)$ is a sequence of subsets $X_t \subseteq C_t$ for every $t \in \{1, \dots, \tau\}$. A committee sequence is valid if $|X_t| \leq k_t$ for every $t \in \{1, \dots, \tau\}$. There are two immediate quality measures for valid committee sequences: the overall utility for an agent and the overall utility for a level. Formally:

$$\mathrm{scr}_{\leftrightarrow}(\mathcal{E}, \mathcal{X}, a) := \sum_{t \in \{1,\dots,\tau\}} \sum_{c \in X_t} u_t(a, c) \quad (\textit{horizontal} \text{ score})$$

$$\text{and } \mathrm{scr}^{\min}_{\leftrightarrow}(\mathcal{E}, \mathcal{X}) := \min_{a \in A} \mathrm{scr}_{\leftrightarrow}(\mathcal{E}, \mathcal{X}, a),$$

$$\mathrm{scr}_{\updownarrow}(\mathcal{E}, \mathcal{X}, t) := \sum_{a \in A} \sum_{c \in X_t} u_t(a, c) \quad (\textit{vertical} \text{ score})$$

$$\text{and } \mathrm{scr}^{\min}_{\updownarrow}(\mathcal{E}, \mathcal{X}) := \min_{t \in \{1,\dots,\tau\}} \mathrm{scr}_{\updownarrow}(\mathcal{E}, \mathcal{X}, t).$$

Deltl et al. [8] introduced a basic version of the model which we generalize (they consider for every $t \in \{1, \dots, \tau\}$ utilities with $z_t = 1$ and $C_t = C$ as well as $k_t = k$ for some given k). In particular when $z_t = 1$ (as in their model) we also call candidates obtaining non-zero utility by an agent *nominated*. Unless specified differently, the utility of a nominated candidate is 1. Deltl et al. [8] studied an immediate computational problem with lower bounds on both scores. One can translate the problem into a rule that selects a committee sequence respecting the bounds. We study—both theoretically and experimentally—further rules which differently fulfill the quality measures, with a focus on the minimum horizontal score.

Our Contributions. We introduce several rules for electing an egalitarian committee sequence. Our five main rules are $\mathcal{R}_{\mathrm{lex}}$, $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$, $\mathcal{R}_{\mathrm{greedy}}$, and $\mathcal{R}_{\mathrm{app}}$ (see Section 3). Intuitively, committee sequences computed by $\mathcal{R}_{\mathrm{lex}}$ firstly maximize the smallest horizontal score, then the second smallest, and so on; $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ firstly maximizes the smallest horizontal score, secondly the sum of horizontal scores, and finally the smallest vertical score; $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ firstly maximizes the smallest horizontal score, then the smallest vertical score,

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[1] We focus on cumulative utilities to have some form of normalization, which is necessary when comparing utilities between different agents.

[2] Note that we could use unordered multisets instead of sequences in our definition, since the actual ordering is not used in rules and properties we focus on. Keeping the definition consistent with the literature, however, allows to keep a simple unified syntax.

and finally the sum of scores. We show that finding a winning committee sequence for these three rules is an NP-hard task. This task is polynomial-time solvable for $\mathcal{R}_{\text{greedy}}$, which intuitively tries to mimic $\mathcal{R}_{\text{lex}}$ by iteratively selecting a candidate that improves $\mathcal{R}_{\text{lex}}$'s objective function the most, and for $\mathcal{R}_{\text{app}}$, which iteratively selects candidates with the highest utility on each level. $\mathcal{R}_{\text{app}}$ is clearly non-egalitarian as it ignores horizontal scores by definition. Yet, we consider $\mathcal{R}_{\text{app}}$ as benchmark, since it mimics the straightforward rule mostly used in practice when customized rules are not available or known. To further evaluate our rules (and distinguish the egalitarian ones), we formulate several properties for rules electing egalitarian committee sequences (see Section 4 for selected central properties). We show that while $\mathcal{R}_{\text{lex}}$ satisfies all of our properties, each of the other four rules violates several desirable properties (see Table 1).

In addition to our theoretical analysis, we investigate the rules experimentally regarding their runtimes, scores, and properties. Each of $\mathcal{R}_{\text{app}}$ and $\mathcal{R}_{\text{greedy}}$ runs faster than the other three, where $\mathcal{R}_{\text{lex}}$ has the highest runtimes on average. When comparing the scores achieved by the rules, we see, e.g., that $\mathcal{R}_{\text{greedy}}$ performs well compared to $\mathcal{R}_{\text{lex}}$ and achieves a clearly better minimum horizontal score on average than $\mathcal{R}_{\text{app}}$ (although we prove that this is not always the case). Interestingly, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ achieves a larger minimum vertical score on average than $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$, whereas $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ has only slightly larger sum of scores. When testing the number of times a rule satisfies the condition of a property that it violates in general, we see, e.g., for each of the properties we present, that $\mathcal{R}_{\text{greedy}}$ satisfies the property's conditions for at least 91% of the instances. Overall, our results indicate that $\mathcal{R}_{\text{greedy}}$ is a good heuristic for $\mathcal{R}_{\text{lex}}$, which in turn is identified as the "most" egalitarian rule.

Related Work. Selecting committees is an important topic from computational social choice with numerous applications following different goals (cf. Elkind et al. [9]) such as individual excellence, proportionality, and diversity. The latter, our focus, is important whenever large parts of agents shall be covered or satisfied, but it is rather hard to formalize (see Section 4.2 from Baumeister et al. [2]). The classical way is to follow an egalitarian approach [1], where the least satisfied agent defines the quality of a committee.

Our model considers multi-level or sequential preferences. Computational aspects of finding sequences of committees have been considered in this context. For example Kellerhals et al. [13] and Bredereck et al. [5] require a minimum satisfaction in each time step with additional constraints on the difference between consecutive committees, but they do not aim for (a minimum) satisfaction of agents. The model we use in this paper was essentially introduced by Deltl et al. [8], who deeply analyzed computational aspects. While distinct formal properties of rules (the focus of our paper) are well-studied in the classical setting [1, 3, 9, 10, 15], we are not aware of any work in the context of (offline) sequential elections. Chandak et al. [7] discuss formal properties for proportional representative online sequences.

Categories (levels) are also used by Boehmer et al. [3], but with the goal of allocating candidates to categories and not selecting from categories (as we do). Other aspects of selecting multiple (sub)committees have been considered for example by Bredereck et al. [4], who also select a sequence of committees. In that work, however, agents have the same preferences for every time step. Mostly focusing on single-winner decisions, Freeman et al. [11], Lackner [14], and Parkes and Procaccia [18] do allow evolving preferences, but in an online setting. More importantly, they use different measures of solution quality. An offline setting is analyzed by Bulteau et al. [6] but aiming for justified representation, which is very different to our egalitarian approach.

Further related to our setting is participatory budgeting (PB), where some community votes on projects. Each project comes with an individual price and there is an overall budget on money being spent. Lackner et al. [16] study a temporal PB setting where agents are partitioned into groups and the goal is to have equal (temporal) fairness after some number of allocation rounds or to optimize the Gini coefficient. Jain et al. [12] study PB with project groups (levels), but aim for optimizing the utilitarian welfare for the voters. Rey et al. [19] design a complex PB framework with the help of judgment aggregation, allowing to model dependencies between projects or quotas for project types, using rules that optimize towards (super-)majorities.

Organization. In Section 3, we introduce and discuss our central rules. In Section 4, we introduce our main properties, explain them, and test them against our central rules. Section 5 shows our experiments. Results marked with $\star$ are deferred to a paper's full version.

2 Preliminaries

We denote by $\mathbb{N}_0$ and $\mathbb{N}$ the set of natural numbers with and without zero, respectively. For two sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$, we denote by $a \circ b$ the sequence $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. For two set sequences $\mathcal{S} = (S_1, \ldots, S_n)$ and $\mathcal{T} = (T_1, \ldots, T_n)$, we denote by $S \cup T$ the sequence $(S_1 \cup T_1, \ldots, S_n \cup T_n)$.

For $A' \subseteq A$, we write $u_t(A', \cdot) \coloneqq \sum_{a \in A'} u_t(a, \cdot)$ for short. Similarly, for $X \subseteq C_t$, we write $u_t(\cdot, X) \coloneqq \sum_{c \in X} u_t(\cdot, c)$ for short. We also combine both notations in the obvious way. For two elections $\mathcal{E}' = (A', \mathcal{C}', U' = (u'_1, \ldots, u'_\tau), \kappa'), \mathcal{E}'' = (A'', \mathcal{C}'', U'' = (u''_1, \ldots, u''_\tau), \kappa'')$ with $A' \cap A'' = \emptyset$, we denote by $U' \cup U''$ the utilities $u_i(a, c) = u'_i(a, c)$ if $a \in A', c \in \mathcal{C}'_i$, $u_i(a, c) = u''_i(a, c)$ if $a \in A'', c \in \mathcal{C}''_i$, and $u_i(a, c) = 0$ otherwise, for $i \in \{1, \ldots, \tau\}$.

Let $\mathbf{E}$ denote the set of all elections. For each $\mathcal{E} = (A, \mathcal{C}, U = (u_1, \ldots, u_\tau), \kappa) \in \mathbf{E}$, let $\mathbf{X}(\mathcal{E}) = \{\mathcal{X} = (X_1, \ldots, X_\tau) \mid \forall t : X_t \subseteq C_t\}$ denote the set of all committee sequences for $\mathcal{E}$. A rule is a mapping $\mathcal{R}(\mathcal{E}) \mapsto \mathbf{X}'$ with $\mathbf{X}' \subseteq \mathbf{X}(\mathcal{E})$. E.g., $\mathcal{R}_{\text{vld}}(\mathcal{E}) \coloneqq \{\text{valid } \mathcal{X} \in \mathbf{X}(\mathcal{E})\} = \{(X_1, \ldots, X_\tau) \in \mathbf{X}(\mathcal{E}) \mid \forall t : |X_t| \le k_t\}$.

For two committee sequences $\mathcal{X}, \mathcal{X}'$ we say that $\mathcal{X}$ $\leftrightarrow$-dominates $\mathcal{X}'$ if for every agent $a \in A$ we have that $\text{scr}_\leftrightarrow(\mathcal{E}, \mathcal{X}, a) \ge \text{scr}_\leftrightarrow(\mathcal{E}, \mathcal{X}', a)$ and for one agent $a' \in A$ we have $\text{scr}_\leftrightarrow(\mathcal{E}, \mathcal{X}, a') > \text{scr}_\leftrightarrow(\mathcal{E}, \mathcal{X}', a')$. We can also define rules upon dominance: $\mathcal{R}_{\leftrightarrow_{\text{dom}}}(\mathcal{E}) \coloneqq \{\mathcal{X} \in \mathcal{R}_{\text{vld}}(\mathcal{E}) \mid \text{there is no } \mathcal{X}' \in \mathcal{R}_{\text{vld}}(\mathcal{E}) \text{ that } \leftrightarrow\text{-dominates } \mathcal{X}\}$.

3 Central Rules

We discuss our central rules $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ and $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ (Section 3.1), $\mathcal{R}_{\text{lex}}$ (Section 3.2), $\mathcal{R}_{\text{greedy}}$ (Section 3.3), and $\mathcal{R}_{\text{app}}$ (Section 3.4).

3.1 Max-Min and Max-Sum Rules

We introduce and study (combinations of) three rules that maximize the minimum horizontal score, the minimum vertical score, and the total sum of the vertical scores (note that this is the same as the sum of the horizontal scores).

An *extended rule* $\mathcal{R}(\mathcal{E}, \mathbf{X}') \mapsto \mathbf{X}''$ with $\mathbf{X}'' \subseteq \mathbf{X}' \subseteq \mathbf{X}(\mathcal{E})$ additionally receives a subset of committee sequences. Let

$$\mathcal{R}_\leftrightarrow(\mathcal{E}, \mathbf{X}') \coloneqq \arg\max_{\mathcal{X} \in \mathbf{X}'} \text{scr}^{\min}_\leftrightarrow(\mathcal{E}, \mathcal{X}), \tag{1}$$

$$\mathcal{R}_\updownarrow(\mathcal{E}, \mathbf{X}') \coloneqq \arg\max_{\mathcal{X} \in \mathbf{X}'} \text{scr}^{\min}_\updownarrow(\mathcal{E}, \mathcal{X}), \text{ and} \tag{2}$$

$$\mathcal{R}_\Sigma(\mathcal{E}, \mathbf{X}') \coloneqq \arg\max_{\mathcal{X} \in \mathbf{X}'} \text{scr}^{\Sigma}_\leftrightarrow(\mathcal{E}, \mathcal{X}), \tag{3}$$

Table 1. Summary of our results. The computational complexity refers to the problem of finding a winning committee sequence. (Superscript at *yes*/*no* refers to the label of the corresponding observation behind the result; *: easy to see)

Property	Rule:	$\mathcal{R}_{\text{lex}}$	$\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$	$\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$	$\mathcal{R}_{\text{greedy}}$	$\mathcal{R}_{\text{app}}$
Computational Complexity		NP-hard	NP-hard	NP-hard	P	P
$\leftrightarrow$-Pareto efficiency (P1)		*yes**	*yes*[5]	*no*[5]	*no*[6]	*yes**
United h-Superadditive (P2)		*yes**	*yes**	*yes**	*no*[7]	*no*[7]
Concatenated h-Superadditive (P3)		*yes**	*yes**	*yes**	*no*[7]	*yes**
Independent Groups (P4)		*yes*[8]	*no*[9]	*no*[9]	*yes*[8]	*yes**
Sub-Consistency (P5)		*yes*[10]	*no*[11]	*no*[11]	*no*[11]	*yes*[10]

where $\text{scr}^{\Sigma}_{\leftrightarrow}(\mathcal{E},\mathcal{X}) := \sum_{a\in A}\text{scr}_{\leftrightarrow}(\mathcal{E},\mathcal{X},a)$ (which is equal to $\text{scr}^{\Sigma}_{\updownarrow}(\mathcal{E},\mathcal{X}) := \sum_{t\in\{1,\dots,\tau\}}\text{scr}_{\updownarrow}(\mathcal{E},\mathcal{X},t)$). We write $\mathcal{R}(\mathcal{E})$ short for $\mathcal{R}(\mathcal{E},\mathbf{X}')$ if $\mathbf{X}' = \mathcal{R}_{\text{vld}}(\mathcal{E})$. To combine the rules in a meaningful way, we better refrain from taking their intersection:

Observation 1 ($\star$). *There is an election $\mathcal{E}$ and $\mathbf{X}' \subseteq \mathbf{X}(\mathcal{E})$ such that $\mathcal{R}_{\updownarrow}(\mathcal{E},\mathbf{X}') \cap \mathcal{R}_{\leftrightarrow}(\mathcal{E},\mathbf{X}') = \emptyset$.*

We combine rules as $\mathcal{R}' \circ \mathcal{R}(\mathcal{E}) = \mathcal{R}'(\mathcal{E},\mathcal{R}(\mathcal{E}))$. For convenience, for a composed rule (we call them generally *max-min max-sum* rules) we list the symbols from $\{\updownarrow,\leftrightarrow,\Sigma\}$ in the order in which the corresponding rules are applied. For instance, we have $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E}) := \mathcal{R}_{\updownarrow} \circ \mathcal{R}_{\Sigma} \circ \mathcal{R}_{\leftrightarrow}(\mathcal{E}) = \mathcal{R}_{\updownarrow}(\mathcal{E},\mathcal{R}_{\leftrightarrow,\Sigma}(\mathcal{E}))$ with $\mathcal{R}_{\leftrightarrow,\Sigma}(\mathcal{E}) := \mathcal{R}_{\Sigma}(\mathcal{E},\mathcal{R}_{\leftrightarrow}(\mathcal{E}))$. The following holds.

Lemma 1 ($\star$). *(i) For every two rules $\mathcal{R},\mathcal{R}'$ and election $\mathcal{E}$ it holds that if $\mathcal{X} \in \mathcal{R}' \circ \mathcal{R}(\mathcal{E})$, then $\mathcal{X} \in \mathcal{R}(\mathcal{E})$. (ii) There are rules $\mathcal{R},\mathcal{R}'$ and an election $\mathcal{E}$ such that $\mathcal{X} \in \mathcal{R}' \circ \mathcal{R}(\mathcal{E})$ but $\mathcal{X} \notin \mathcal{R}'(\mathcal{E})$. (iii) $\circ$ is not commutative.*

We next show how $\mathcal{R}_{\updownarrow}$, $\mathcal{R}_{\Sigma}$, and their compositions relate. Then, we settle all combinations' computational complexity.

Lemma 2 ($\star$). *For every election $\mathcal{E}$ it holds true that $\mathcal{R}_{\updownarrow}(\mathcal{E}) \supseteq \mathcal{R}_{\Sigma}(\mathcal{E}) = \mathcal{R}_{\updownarrow,\Sigma}(\mathcal{E}) = \mathcal{R}_{\Sigma,\updownarrow}(\mathcal{E})$.*

Theorem 1 ($\star$). *(i) The problem of finding a winning committee sequence for any combination of $\mathcal{R}_{\Sigma}$ and $\mathcal{R}_{\updownarrow}$ is polynomial-time solvable. (ii) The problem of finding a winning committee sequence for any max-min max-sum rule containing $\mathcal{R}_{\leftrightarrow}$ is NP-hard.*

We focus on $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ and $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ since they are egalitarian (maximizing the minimum horizontal score first). Despite Lemma 2, they are different (also in practice, as we see later).

Observation 2 ($\star$). $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E}) \neq \mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}(\mathcal{E})$ *for some $\mathcal{E}$.*

3.2 Lex Rule

We rate our next rule $\mathcal{R}_{\text{lex}}$ as the "most egalitarian" of our rules: it not only maximizes the minimum horizontal score, but also minimizes the number of agents with small scores.

$$\mathcal{R}_{\text{lex}}(\mathcal{E}) := \arg\min_{\mathcal{X}\in\mathcal{R}_{\text{vld}}(\mathcal{E})} \text{ems}(\mathcal{E},\mathcal{X}) \tag{4}$$

with $\text{ems}(\mathcal{E},\mathcal{X}) := \sum_{i=0}^{Z}\text{sat}(\mathcal{E},\mathcal{X},i)\cdot(|A|+1)^{Z-i}$ with $Z := \sum_{t\in\{1,\dots,\tau\}} z_t$ and $\text{sat}(\mathcal{E},\mathcal{X},i) := |\{a \in A \mid \text{scr}_{\leftrightarrow}(\mathcal{E},\mathcal{X},a) = i\}|$. $\mathcal{R}_{\text{lex}}$ maximizes lexicographically the agents' satisfaction histogram $\overrightarrow{\text{sat}}(\mathcal{E},\mathcal{X}) = (\text{sat}(\mathcal{E},\mathcal{X},0),\text{sat}(\mathcal{E},\mathcal{X},1),\dots,\text{sat}(\mathcal{E},\mathcal{X},Z))$. Here, we lexicographically order in the standard way: For two number sequences $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$, we write $x \prec y$ if for the smallest index i where x and y differ we have that $x_i < y_i$. We have the following, well-known connection.

Lemma 3 ($\star$). *For every election $\mathcal{E}$ and committee sequences $\mathcal{X},\mathcal{X}' \in \mathbf{X}(\mathcal{E})$ it holds true that $\text{ems}(\mathcal{E},\mathcal{X}) < \text{ems}(\mathcal{E},\mathcal{X}') \iff \overrightarrow{\text{sat}}(\mathcal{E},\mathcal{X}) \prec \overrightarrow{\text{sat}}(\mathcal{E},\mathcal{X}')$.*

Obviously, $\mathcal{R}_{\text{lex}}(\mathcal{E}) \subseteq \mathcal{R}_{\leftrightarrow}(\mathcal{E})$ for every election $\mathcal{E}$ and hence by Theorem 1(ii):

Fact 1. *The problem of finding a winning committee sequence for $\mathcal{R}_{\text{lex}}$ is NP-hard.*

$\mathcal{R}_{\text{lex}}$ is different from $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ and $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ (and $\mathcal{R}_{\leftrightarrow_{\text{dom}}}$).

Observation 3 ($\star$). *(i) There is an election $\mathcal{E}$ such that $\mathcal{R}_{\text{lex}}(\mathcal{E}) \setminus (\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E}) \cup \mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}(\mathcal{E})) \neq \emptyset$ and $(\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E}) \cap \mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}(\mathcal{E})) \setminus \mathcal{R}_{\text{lex}}(\mathcal{E}) \neq \emptyset$. (ii) We have $\mathcal{R}_{\text{lex}}(\mathcal{E}) \subseteq \mathcal{R}_{\leftrightarrow_{\text{dom}}}(\mathcal{E})$ for every election $\mathcal{E}$. (iii) There exists $\mathcal{E}'$ such that $\mathcal{R}_{\leftrightarrow_{\text{dom}}}(\mathcal{E}') \setminus \mathcal{R}_{\text{lex}}(\mathcal{E}') \neq \emptyset$.*

3.3 Greedy Rule

Since finding a winning committee sequence for our three preceding egalitarian rules is NP-hard, we present a polynomial-time greedy rule: On each level t, the initial solution selects all candidates receiving positive utility by at least one agent if there are exactly k_t many of them, and the empty set otherwise. Then the rule repeats the following: Among all levels where selecting a candidate is possible (i.e., the current committee on level t is smaller than k_t) add to the committee of the respective level one of the candidates not yet selected whose decrease of the ems score of the current solution is largest.

Definition 2. Rule $\mathcal{R}_{\text{greedy}}$ is formally defined as follows.

1. For each $t \in \{1,\dots,\tau\}$, let $C^n_t := \{c \in C_t \mid u_t(A,c) > 0\}$ and $k_t := \min\{k_t, |C^n_t|\}$. Start with $\mathcal{X} = (X_1,\dots,X_\tau)$, where $X_t = C^n_t$ if $|C^n_t| = k_t$, else $X_t = \emptyset$, for each $t \in \{1,\dots,\tau\}$.
2. Let $T' := \{t \in \{1,\dots,\tau\} \mid |X_t| < k_t\}$. If $|T'| = 0$, stop.
3. Pick some $(t',c') \in \arg\min_{(t,c)\in S}\text{ems}(\mathcal{E},\mathcal{X} \cup_t \{c\})$, where $S = \{(t,c) \mid t \in T', c \in C^n_t \setminus X_t\}$ and $\mathcal{X} \cup_t \{c\} := (X_1,\dots,X_t \cup \{c\},\dots,X_\tau)$. Add this c' to $X_{t'}$. Go to step 2.

Definition 2 also describes how to find a committee sequence winning for $\mathcal{R}_{\text{greedy}}$ in polynomial time. Hence:

Fact 2. *The problem of finding a winning committee sequence for $\mathcal{R}_{\text{greedy}}$ is polynomial-time solvable.*

3.4 Approval Rule

Our last rule selects committees in a very obvious way: independently for each level, select iteratively the candidate that receives the hightest utility. This is clearly not an egalitarian rule. Let $\mathcal{P}^*_t(\mathcal{E}) := \arg\max_{X'\subseteq C_t\colon |X'|\le k_t} u_t(A,X')$ and

$$\mathcal{R}_{\text{app}}(\mathcal{E}) := \mathcal{P}^*_1(\mathcal{E}) \times \cdots \times \mathcal{P}^*_\tau(\mathcal{E}). \tag{5}$$

We have already seen the winning set of $\mathcal{R}_{\text{app}}$: For every election $\mathcal{E}$ it holds true that $\mathcal{R}_{\text{app}}(\mathcal{E}) = \mathcal{R}_{\Sigma}(\mathcal{E})$. The following is thus also immediate (see also Theorem 1(i)):

Fact 3. *The problem of finding a winning committee sequence for $\mathcal{R}_{\text{app}}$ is polynomial-time solvable.*

Although $\mathcal{R}_{\text{greedy}}$ in each step treats the least-satisfied agent that still can improve, its outcome could be "less" egalitarian than one by our non-egalitarian rule $\mathcal{R}_{\text{app}}$.

Observation 4 ($\star$)**.** *There is an election $\mathcal{E}$ and two committee sequences $\mathcal{X} \in \mathcal{R}_{\text{app}}(\mathcal{E})$ and $\mathcal{X}' \in \mathcal{R}_{\text{greedy}}(\mathcal{E})$ such that $\text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}) > \text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}')$.*

4 Central Properties

In this section, we discuss the fingerprints of our rules on the selected central properties (see Table 1 for an overview). Exemplarily, we provide here some selected proofs. We defer the complete theoretical analysis (with many more properties) to a full version of the paper.

We introduce, explain, and define our main properties against which we test our rules. We find the selected properties particularly well-motivated for the goal of finding egalitarian committee sequences. Moreover, no two properties are satisfied by the same set of rules and we can distinguish each pair of rules with one of the properties.

Pareto efficiency. The first property is based on the concept of domination as defined in Section 2. When aiming for satisfaction of individual (in particular, least satisfied) agents, providing a $\leftrightarrow$-dominated committee sequence is a bad idea. Thus, we are interested in rules that are $\leftrightarrow$-Pareto efficient as defined next.

Property 1 ($\leftrightarrow$-Pareto efficiency)**.** A rule satisfies $\leftrightarrow$-Pareto efficiency if it outputs no committee sequence which is $\leftrightarrow$-dominated by a valid committee sequence.

By Observation 3(ii), $\mathcal{R}_{\text{lex}}$ satisfies *$\leftrightarrow$-Pareto efficiency (P1)*. For the max-min max-sum rules we have the following.

Observation 5. *Every rule composed of $\mathcal{R}_{\leftrightarrow}$, $\mathcal{R}_{\updownarrow}$, and $\mathcal{R}_{\Sigma}$ containing $\mathcal{R}_{\Sigma}$ satisfies $\leftrightarrow$-Pareto efficiency (P1) except for $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\updownarrow,\leftrightarrow,\Sigma}$ (they violate $\leftrightarrow$-Pareto efficiency (P1)).*

Proof. For first statement, note that every addressed composition contains $\mathcal{R}_{\Sigma}$ as first or second rule. By definition, we have that if $\mathcal{X}$ $\leftrightarrow$-dominates $\mathcal{X}'$, then $\text{scr}^{\Sigma}_{\leftrightarrow}(\mathcal{E},\mathcal{X}) > \text{scr}^{\Sigma}_{\leftrightarrow}(\mathcal{E},\mathcal{X}')$. Hence, the statement follows from the definition of $\mathcal{R}_{\Sigma}$ together with Lemma 2.

For the second statement, see Figure 1(i).[3] □

It follows that $\mathcal{R}_{\text{app}}$ satisfies *$\leftrightarrow$-Pareto efficiency (P1)*. This is not the case for $\mathcal{R}_{\text{greedy}}$.

Observation 6 (Proof by Figure 1(ii))**.** *$\mathcal{R}_{\text{greedy}}$ violates $\leftrightarrow$-Pareto efficiency (P1).*

Superadditivity. The next two properties capture situations where two elections are "merged" in two different ways. First, we model situations (*united*) where two disjoint groups of agents vote over the same levels. Second, we model situations (*concatenated*) where the same group of agents votes over two different disjoint sets of levels. In both cases, our property is based on the idea that in a merged election, the satisfaction of the least satisfied agent(s) shall not get worse.

[3] All our counter examples use $\{0,1\}$-utilities which we represent as nominations $u_t\colon A \to C_t \cup \{\emptyset\}$, $t \in \{1,\dots,\tau\}$, for the sake of readability, where in each level each agent nominates either none or her favorite candidate.

(i)

a_1:	a	$\boxed{c}$	–
a_2:	a	$\boxed{c}$	$\boxed{f}$
a_3:	–	$\boxed{c}$	e
a_4:	–	$\boxed{c}$	e
a_5:	$\boxed{b}$	d	–
a_6:	–	d	$\boxed{f}$

(ii)

a_1:	a	c	$\boxed{e}$
a_2:	a	–	$\boxed{e}$
a_3:	$\boxed{b}$	–	f
a_4:	–	c	$\boxed{e}$
a_5:	–	$\boxed{d}$	f

Figure 1. Counterexample to *$\leftrightarrow$-Pareto efficiency (P1)* for (i) $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ with $\kappa = (1,1,1)$. Note that $\mathcal{X} = (\{a\},\{d\},\{e\})$ (blue circle) and $\mathcal{X}' = (\{b\},\{c\},\{f\})$ (green square) are the only valid committee sequences with minimum horizontal score of one (clearly, $\text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}) \le 1$ for every valid $\mathcal{X}$). Since $2 = \text{scr}^{\min}_{\updownarrow}(\mathcal{E},\mathcal{X}) > \text{scr}^{\min}_{\updownarrow}(\mathcal{E},\mathcal{X}') = 1$, each of $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\updownarrow,\leftrightarrow,\Sigma}$ selects $\mathcal{X}$. However, $\mathcal{X}'$ $\leftrightarrow$-dominates $\mathcal{X}$. (ii) $\mathcal{R}_{\text{greedy}}$ with $\kappa = (1,1,1)$. $\mathcal{R}_{\text{greedy}}$ selects $(\{b\},\{d\},\{e\})$ (green square), which is $\leftrightarrow$-dominated by $(\{a\},\{c\},\{f\})$ (blue circle).

Property 2 (United h-Superadditive)**.** Rule $\mathcal{R}$ is *united h-superadditive* if for any two elections $\mathcal{E}_1 = (A_1,\mathcal{C}_1,U_1,\kappa_1)$ and $\mathcal{E}_2 = (A_2,\mathcal{C}_2,U_2,\kappa_2)$ over the same number of levels but disjoint agent sets we have that for all $\mathcal{X}_1 \in \mathcal{R}(\mathcal{E}_1), \mathcal{X}_2 \in \mathcal{R}(\mathcal{E}_2)$ there is an $\mathcal{X} \in \mathcal{R}(\mathcal{E})$ with $\mathcal{E} = (A_1 \cup A_2, \mathcal{C}_1 \cup \mathcal{C}_2, U_1 \cup U_2, \kappa_1 + \kappa_2)$ such that $\text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}) \ge \text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}_1 \cup \mathcal{X}_2)$.

Property 3 (Concatenated h-Superadditive)**.** Rule $\mathcal{R}$ is *concatenated h-superadditive* if for every two elections $\mathcal{E}_1 = (A,\mathcal{C},U,\kappa)$ and $\mathcal{E}_2 = (A,\mathcal{C}',U',\kappa')$ we have that for all $\mathcal{X}_1 \in \mathcal{R}(\mathcal{E}_1), \mathcal{X}_2 \in \mathcal{R}(\mathcal{E}_2)$ there is an $\mathcal{X} \in \mathcal{R}(\mathcal{E})$ with $\mathcal{E} = (A, \mathcal{C} \circ \mathcal{C}', U \circ U', \kappa \circ \kappa')$ so that $\text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}) \ge \text{scr}^{\min}_{\leftrightarrow}(\mathcal{E},\mathcal{X}_1 \circ \mathcal{X}_2)$.

Since $\mathcal{R}_{\text{lex}}$, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$, and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ maximize the minimum horizontal score, they trivially satisfy *United h-Superadditive (P2)* and *Concatenated h-Superadditive (P3)*. $\mathcal{R}_{\text{app}}$ also trivially satisfies *Concatenated h-Superadditive (P3)* since the concatenation of two winning committee sequences is again winning. Less obvious is the situation for $\mathcal{R}_{\text{app}}$ regarding *United h-Superadditive (P2)* and for $\mathcal{R}_{\text{greedy}}$.

Observation 7 (Proof by Figure 2)**.** *$\mathcal{R}_{\text{greedy}}$ violates United h-Superadditive (P2) and Concatenated h-Superadditive (P3). $\mathcal{R}_{\text{app}}$ violates United h-Superadditive (P2).*

Independent Groups. While it is desired that additional levels or agents will influence the outcome (in order to maximize satisfaction of the least satisfied agents), the next property basically says that when different agent groups have disjoint interests, then the outcome should be the same as if each agent group attended a separate election.

Property 4 (Independent Groups)**.** Rule $\mathcal{R}$ satisfies *independent groups* if for every $\mathcal{E} = (A, \mathcal{C} = (C_1,\dots,C_\tau), U = (u_1,\dots,u_\tau), \kappa = (k_1,\dots,k_\tau))$ with $A = A_1 \uplus \dots \uplus A_r$ and $1 \le t_1 < t_2 < \dots < t_{r-1} < \tau = t_r$ such that among all levels $t \in \{t_{s-1}+1,\dots,t_s\} =: T_s$, where $t_0 = 0$, exactly the group A_s has positive utilities, it holds true that $\mathcal{R}(\mathcal{E}) = \mathcal{R}(\mathcal{E}_1) \times \dots \times \mathcal{R}(\mathcal{E}_r)$, where $\mathcal{E}_s = (A_s, (C_t)_{t\in T_s}, (u_t)_{t\in T_s}, (k_t)_{t\in T_s})$, $s \in \{1,\dots,r\}$, are the corresponding elections.

$\mathcal{R}_{\text{app}}$ trivially satisfies *Independent Groups (P4)* since it selects committees for each level independently. Intuitively, ems also optimizes each group independently and, indeed, we have the following.

Observation 8. *Each of (i) $\mathcal{R}_{\text{lex}}$ and (ii) $\mathcal{R}_{\text{greedy}}$ satisfies Independent Groups (P4).*

Proof. (i) Let $\mathcal{X} \in \mathcal{R}_{\text{lex}}(\mathcal{E})$. Let $\mathcal{X} = \bigcup_{s=1}^{r} \mathcal{X}_s$, where $\mathcal{X}_s$ has only non-empty committees on T_s, and let $\bar{\mathcal{X}}_s$ be $\mathcal{X}_s$ restricted to the

(i)

a_1:	a	c	e	g
a_2:	a	c	f	h
a_3:	b	d	e	$-$

(ii)

a_1:	a	d
a_2:	a	d
a_3:	a	e
a_4:	b	e
a'_1:	b	f
a'_2:	c	$-$

(iii)

a_1:	a	d
a_2:	a	d
a_3:	a	e
a_4:	b	e
a_5:	b	e
a'_1:	c	f

Figure 2. Counterexample to (i) *Concatenated h-Superadditive (P3)* for $\mathcal{R}_{\text{greedy}}$ with $\kappa = (1, \dots, 1)$. After concatenation (along the dashed line), $\mathcal{R}_{\text{greedy}}$ selects the committee $(\{a\}, \{c\}, \{e\}, \{h\})$ (green square). This leaves a_3 with a horizontal score of only one, although the concatenation of individual winners (blue circle) provides each agent with a horizontal score of two. (ii) *United h-Superadditive (P2)* for $\mathcal{R}_{\text{greedy}}$ with $\mathcal{E}_1$ (top) and $\mathcal{E}_2$ (bottom), separated by a dashed line, with $\kappa = (1, 1)$ each. Note that in the united election with $\kappa' = (2, 2)$, $\mathcal{R}_{\text{greedy}}$ selects a first, then b, and then d and e, leaving agent a'_2 with score zero (green square). However, $(\{a\}, \{e\}) \in \mathcal{R}_{\text{greedy}}(\mathcal{E}_1)$ together with $(\{c\}, \{f\}) \in \mathcal{R}_{\text{greedy}}(\mathcal{E}_2)$ satisfies every agent at least once (blue circle). (iii) *United h-Superadditive (P2)* for $\mathcal{R}_{\text{app}}$ with $\mathcal{E}_1$ (top) and $\mathcal{E}_2$ (bottom), separated by a dashed line, with $\kappa = (1, 1)$ each. Note that $\mathcal{R}_{\text{app}}(\mathcal{E}_1) = (\{a\}, \{e\})$ and $\mathcal{R}_{\text{app}}(\mathcal{E}_2) = (\{c\}, \{f\})$, leaving each agent satisfied at least once (blue circle). However, in the united election $\mathcal{E}$ with $\kappa' = (2, 2)$, $\mathcal{R}_{\text{app}}(\mathcal{E}) = (\{a, b\}, \{d, e\})$ (green square), leaving agent a'_1 unsatisfied.

levels in T_s. For every $A' \subseteq A$ and committee sequence $\mathcal{X}' \in \mathcal{R}_{\text{vld}}(\mathcal{E})$, let $\text{sat}(\mathcal{X}', A', i) := |\{a \in A' \mid \text{scr}_{\leftrightarrow}(\mathcal{E}, \mathcal{X}', a) = i\}|$ and $\overrightarrow{\text{sat}}(\mathcal{X}', A') := (\text{sat}(\mathcal{X}', A', 0), \dots, \text{sat}(\mathcal{X}', A', Z))$. We have $\overrightarrow{\text{sat}}(\mathcal{X}, A) = \sum_{s=1}^{r} \overrightarrow{\text{sat}}(\mathcal{X}_s, A_s)$. Suppose there is an $\bar{\mathcal{X}}'_s \in \mathcal{R}_{\text{lex}}(\mathcal{E}_s)$ with $\text{ems}(\mathcal{E}_s, \bar{\mathcal{X}}'_s) < \text{ems}(\mathcal{E}_s, \bar{\mathcal{X}}_s)$, which is equivalent to $\overrightarrow{\text{sat}}(\mathcal{E}_s, \bar{\mathcal{X}}'_s) \prec \overrightarrow{\text{sat}}(\mathcal{E}_s, \bar{\mathcal{X}}_s)$ (see Lemma 3). The latter is equivalent to $\overrightarrow{\text{sat}}(\mathcal{X}'_s, A_s) \prec \overrightarrow{\text{sat}}(\mathcal{X}_s, A_s)$, where $\mathcal{X}'_s$ is $\bar{\mathcal{X}}'_s$ with $\sum_{q=1}^{s-1} |T_q|$ preceding and $\sum_{q=s+1}^{r} |T_q|$ succeeding empty committees. Thus, $\mathcal{X}'$, where we replace $\mathcal{X}_s$ with $\mathcal{X}'_s$, has $\overrightarrow{\text{sat}}(\mathcal{X}', A) \prec \overrightarrow{\text{sat}}(\mathcal{X}, A)$ and hence $\text{ems}(\mathcal{E}, \mathcal{X}') < \text{ems}(\mathcal{E}, \mathcal{X})$ (see Lemma 3), a contradiction to $\mathcal{X} \in \mathcal{R}_{\text{lex}}(\mathcal{E})$. The other directions goes analogously.

The proof of (ii) is deferred to the full version. $\square$

The situation is different for $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ and $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$: a group with small minimum horizontal score can lead to optimizing the sum of scores for another group among committees with minimum horizontal score lower than what the group could achieve.

Observation 9 (Proof by Figure 3)**.** *Each of* $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ *and* $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ *violates Independent Groups (P4).*

a_1:	x	$-$	$-$	$-$	$-$
a_2:	$-$	a	a	a	a
a_3:	$-$	b	b	b	b
a_4:	$-$	b	b	b	b

Figure 3. Counterexamples for $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ (also works for $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$) and *Independent Groups (P4)* with $\kappa = (1, \dots, 1)$ and the two elections $\mathcal{E}_1$ and $\mathcal{E}_2$ indicated by dashed boxes. We have that $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E}_1) = \{x\}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E}_2)$ selects each of candidate a and b exactly twice (blue circle). $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}(\mathcal{E})$, however, selects a only once (green square).

Consistency. The next property was also adapted by Boehmer et al. [3] for "line-up elections" (also appears in [15]). If we have two agent groups interested in the same levels and separate elections for them share a common solution, then this solution should also be a solution for the elections' union. This is practical since in this case merging two groups with a common solution then makes the recomputation of a winning committee sequence optional.

Property 5 (Sub-Consistency)**.** Rule $\mathcal{R}$ satisfies *sub-consistency* if for every two elections $\mathcal{E}_1 = (A_1, \mathcal{C}, U_1, \kappa)$ and $\mathcal{E}_2 = (A_2, \mathcal{C}, U_2, \kappa)$ over the same candidate sequence $\mathcal{C}$, number of levels, and committee bounds, but disjoint agent sets A_1 and A_2, it holds that if $M := \mathcal{R}(\mathcal{E}_1) \cap \mathcal{R}(\mathcal{E}_2) \neq \emptyset$, then $M \subseteq \mathcal{R}(A_1 \cup A_2, \mathcal{C}, U_1 \cup U_2, \kappa)$.

$\mathcal{R}_{\text{lex}}$ and $\mathcal{R}_{\text{app}}$ satisfy the property as the only two of the five rules.

Observation 10. *Each of (i)* $\mathcal{R}_{\text{lex}}$ *and (ii)* $\mathcal{R}_{\text{app}}$ *satisfies Sub-Consistency (P5).*

Proof. (i): Let $B \in \{A_1, A_2, A\}$ with $A = A_1 \cup A_2$ and $\text{ems}_B(\mathcal{X}) := \sum_{i=0}^{Z} |\{a \in B \mid \text{scr}_{\leftrightarrow}(\mathcal{E}, \mathcal{X}, a) = i\}| \cdot (|A| + 1)^{Z-i}$ with $\mathcal{E} = (A, \mathcal{C}, U_1 \cup U_2, \kappa)$. It holds that $\text{ems}_A(\mathcal{X}) = \text{ems}(\mathcal{E}, \mathcal{X})$. We use the following: $\text{ems}(\mathcal{E}_i, \mathcal{X}_1) \leq \text{ems}(\mathcal{E}_i, \mathcal{X}_2) \iff \text{ems}_{A_i}(\mathcal{X}_1) \leq \text{ems}_{A_i}(\mathcal{X}_2)\ \forall i \in \{1, 2\}$ $(*)$. We show that $M \subseteq \mathcal{R}_{\text{lex}}(\mathcal{E})$. Let $\mathcal{X}_M \in M$. Then $\min_{\mathcal{X}} \text{ems}_A(\mathcal{X}) = \min_{\mathcal{X}} (\text{ems}_{A_1}(\mathcal{X}) + \text{ems}_{A_2}(\mathcal{X})) \geq \min_{\mathcal{X}} \text{ems}_{A_1}(\mathcal{X}) + \min_{\mathcal{X}} \text{ems}_{A_2}(\mathcal{X}) \overset{(*)}{=} \text{ems}_{A_1}(\mathcal{X}_M) + \text{ems}_{A_2}(\mathcal{X}_M) = \text{ems}_A(\mathcal{X}_M)$ and thus $\mathcal{X} \in \mathcal{R}_{\text{lex}}(\mathcal{E})$.

(ii): Let $\mathcal{E}_i = (A_i, \mathcal{C}, U_i = (u_{i,1}, \dots, u_{i,\tau}), \kappa)$ for each $i \in \{1, 2\}$ and let $\mathcal{E} = (A_1 \cup A_2, \mathcal{X}, U_1 \cup U_2 = (u_1, \dots, u_\tau), \kappa)$. Let $\mathcal{X} = (X_1, \dots, X_\tau) \in M$. Assume that $\mathcal{X} \notin \mathcal{R}_{\text{app}}(\mathcal{E})$. Then there is a $t \in \{1, \dots, \tau\}$ such that $X_t \notin P^*_t(\mathcal{E})$. Let $X'_t \in P^*_t(\mathcal{E})$. We have that $u_{1,t}(A, X_t) + u_{2,t}(A, X_t) < u_t(A, X'_t) = u_{1,t}(A, X'_t) + u_{2,t}(A, X'_t)$, and thus $u_{1,t}(A, X_t) < u_{1,t}(A, X'_t)$ or $u_{2,t}(A, X_t) < u_{2,t}(A, X'_t)$, which contradicts the choice of $\mathcal{X}$. $\square$

Observation 11 (Proof by Figure 4)**.** *Each of* $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$, *and* $\mathcal{R}_{\text{greedy}}$ *violates Sub-Consistency (P5).*

(i)

a_1:	a	c	a'_1:	b	b
a_2:	b	a	a'_2:	a	a
a_3:	b	b	a'_3:	a	a
a_4:	a	b	a'_4:	a	a

(ii)

a_1:	a	c	e
a_2:	a	c	f
a'_1:	b	c	f
a'_2:	a	d	e
a'_3:	a	$-$	e

Figure 4. Counterexample to *Sub-Consistency (P5)* for (i) $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ and $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ with $\kappa = (1, 1)$. We have that $(\{a\}, \{b\})$ (blue circle) is the only common winner for $\mathcal{E}_1$ (left) and $\mathcal{E}_2$ (right). For their union, note that $(\{a\}, \{a\})$ (green square) is the only winner. (ii) $\mathcal{R}_{\text{greedy}}$ with $\kappa = (1, 1, 1)$. A common winner is $(\{a\}, \{c\}, \{f\})$ (blue circle): For $\mathcal{E}_2$ (bottom), selecting first a, then f, and then c is possible. For the union, the only winner is $(\{a\}, \{c\}, \{e\})$ (green square).

5 Experiments

We analyze the behavior of $\mathcal{R}_{\text{lex}}$, $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$, $\mathcal{R}_{\text{greedy}}$, and $\mathcal{R}_{\text{app}}$ when applied to experimental data, which is discussed in Section 5.1, using the experimental setup discussed in Section 5.2. Based on this, multiple aspects of the rules' behavior are discussed in subsequent sections: The values of different scores of the winning committee sequences determined by different rules (Section 5.3); the runtimes of the rules when applied to the experimental data (Section 5.4); and the proportion of experimental elections for which a given rule satisfies a given property from Section 4 (Section 5.5). We consider $\kappa = (1, \dots, 1)$, since it is the simplest non-trivial case (cf. [8]) and the qualitative results regarding the scores only change a little overall for other κ that we have tested.

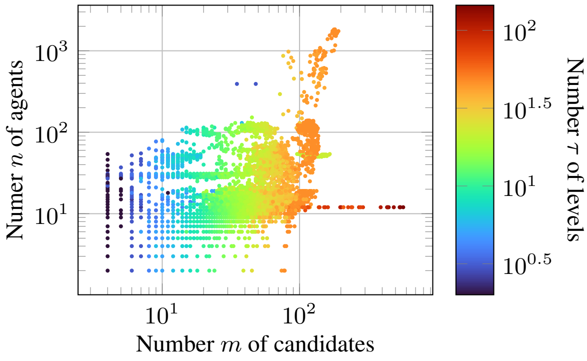

Figure 5. The dimensions of the experimental data, where the color of each point represents the average τ of all instances with the given m and n.

5.1 *Experimental Data*

To the best of our knowledge, there are no (real-life) datasets for electing sequences of committees. Thus, we transformed preference datasets from PrefLib [17], where preferences are represented as strict (possibly incomplete) linear orders, into nomination profiles[4] as follows. Based on the respective metadata, we chose levels and assigned the candidates accordingly (a candidate may be assigned to multiple levels). On each level, each agent nominates its most preferred candidate if any candidate on this level is in the agent's preference order, otherwise it nominates none. Using this approach, we have created a total of 8870 sequential elections with the following additional steps: We removed candidates not nominated by any agent and levels on which all agents nominate the same candidate. After this, we removed instances with at most one level or three candidates. Figure 5 gives an overview of the number of agents, levels, and candidates in this experimental data.

5.2 *Experimental Setup*

The sequential election rules have been implemented in Python to analyze their behavior. Additionally, $\mathcal{R}_{\text{greedy}}$ has also been implemented in C++ to compute all winning committee sequences in Section 5.5. While the implementations of $\mathcal{R}_{\text{greedy}}$ and $\mathcal{R}_{\text{app}}$ follow directly from their definitions in Section 3.3 and Section 3.4, respectively, the other rules are defined as constraint optimization problems (CPs) and implemented as well as solved using the CP-SAT solver from Google OR-Tools (version 9.8.3296 with default parameters).

Outside Section 5.5, we only consider one winning committee sequence, even if there is more than one, for every rule and election. For $\mathcal{R}_{\text{lex}}$, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$, and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$, the first winning committee sequence found by the CP-SAT solver is considered. For $\mathcal{R}_{\text{greedy}}$ and $\mathcal{R}_{\text{app}}$, the first possible candidate in the candidate order of the underlying PrefLib dataset is chosen.

5.3 *The Scores*

One interesting aspect is how well rules designed to optimize one set of scores behave with respect to other scores. Table 2 shows a comparison of the scores of the winning committee sequences of our rules on the experimental data.

[4] Experiments with more complex utilities are ongoing work, but require very extensive and time-consuming systematic analysis. Preliminary findings with Borda-like utilities support all main conclusions made here.

Table 2. The first value is the percentage of the experimental data for which a rule achieves the optimal score. The value in parentheses is the percentage of the optimal score that a rule achieves on average, with one exception: For ems, the percentage of the reciprocal proportion is shown, as ems is the only score to be minimized. If a rule is optimal with respect to a score, then the corresponding cell contains 100. The percentages are rounded naturally.

Rule	ems	$\text{scr}^{\min}_{\leftrightarrow}$	$\text{scr}^{\min}_{\updownarrow}$	$\text{scr}^{\Sigma}_{\leftrightarrow}$
$\mathcal{R}_{\text{lex}}$	100	100	27 (60)	13 (93)
$\mathcal{R}_{\text{greedy}}$	61 (86)	90 (99)	27 (60)	14 (93)
$\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$	45 (83)	100	38 (69)	23 (95)
$\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$	43 (82)	100	39 (72)	23 (95)
$\mathcal{R}_{\text{app}}$	10 (20)	19 (73)	100	100

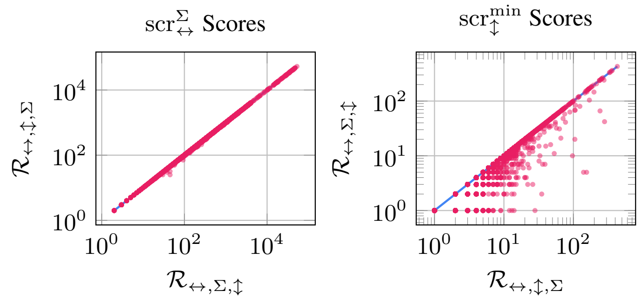

Figure 6. A comparison of $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$, where each point represents one instance from the experimental data. On the left (right), the x value represents the $\text{scr}^{\Sigma}_{\leftrightarrow}$ score ($\text{scr}^{\min}_{\updownarrow}$ score) achieved by $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ ($\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$) and the y value the same score achieved by $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ ($\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$). If a point is on the blue line, the scores reached by the two rules are the same.

Comparing $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$. Besides satisfying the same properties from Section 4 apart from one, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ behave similarly with respect to the optimal score on average. However, these two rules achieve different $\text{scr}^{\Sigma}_{\leftrightarrow}$ and $\text{scr}^{\min}_{\updownarrow}$ scores for around 11% of the instances and can differ significantly (see Figure 6). While $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ reaches very similar $\text{scr}^{\Sigma}_{\leftrightarrow}$ scores to $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ with very few outliers, there are significant outliers regarding the $\text{scr}^{\min}_{\updownarrow}$ score: There are instances for which $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ achieves only around 3% of the $\text{scr}^{\min}_{\updownarrow}$ score achieved by $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$. Nevertheless, both rules reach better $\text{scr}^{\min}_{\updownarrow}$ scores than $\mathcal{R}_{\text{lex}}$ and $\mathcal{R}_{\text{greedy}}$ on average.

Table 2 also shows that, interestingly, the optimal ems score is 82% and 83% of the score reached by $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ on average, respectively, despite neither rule optimizing ems .

Comparing $\mathcal{R}_{\text{lex}}$, $\mathcal{R}_{\text{greedy}}$, and $\mathcal{R}_{\text{app}}$. The $\text{scr}^{\min}_{\leftrightarrow}$ scores of $\mathcal{R}_{\text{app}}$, $\mathcal{R}_{\text{greedy}}$, and $\mathcal{R}_{\text{lex}}$ are compared in the scatter plot on the left of Figure 7: $\mathcal{R}_{\text{greedy}}$ achieves 99% of the optimal $\text{scr}^{\min}_{\leftrightarrow}$ score on average, has no severe outliers, and reaches the exact optimal $\text{scr}^{\min}_{\leftrightarrow}$ score for 90% of the experimental data. $\mathcal{R}_{\text{app}}$ exhibits severe outliers regarding the $\text{scr}^{\min}_{\leftrightarrow}$ score, achieving 0% of the optimal score for the worst outliers. With respect to the average ems score, $\mathcal{R}_{\text{greedy}}$ outperforms every other rule (except $\mathcal{R}_{\text{lex}}$), achieving an ems score that is only about 1.2 times as large as the optimum on average, as opposed to, e.g., five times in the case of $\mathcal{R}_{\text{app}}$. Similarly, $\mathcal{R}_{\text{greedy}}$ reaches the optimal score for about 61% of the experimental data, while all other rules (except $\mathcal{R}_{\text{lex}}$) achieve it for less than 50% of the data, with $\mathcal{R}_{\text{app}}$ being the worst with only 10%. This demonstrates that $\mathcal{R}_{\text{greedy}}$ is a far better choice than $\mathcal{R}_{\text{app}}$ with respect to $\text{scr}^{\min}_{\leftrightarrow}$ and ems.

However, $\mathcal{R}_{\text{app}}$ does not have as many outliers regarding the $\text{scr}^{\min}_{\leftrightarrow}$ score as $\mathcal{R}_{\text{lex}}$ and $\mathcal{R}_{\text{greedy}}$ regarding the $\text{scr}^{\min}_{\updownarrow}$ score, as shown by

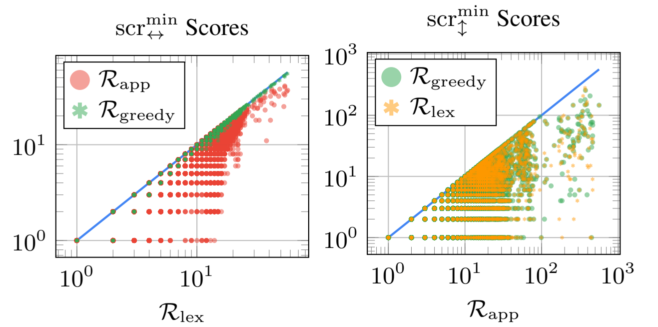

Figure 7. A comparison of $\mathcal{R}_{\mathrm{lex}}$, $\mathcal{R}_{\mathrm{greedy}}$, and $\mathcal{R}_{\mathrm{app}}$: Each point represents one instance from the experimental data. On the left (right), the x value represents the $\mathrm{scr}^{\min}_{\leftrightarrow}$ ($\mathrm{scr}^{\min}_{\updownarrow}$) score achieved by $\mathcal{R}_{\mathrm{lex}}$ ($\mathcal{R}_{\mathrm{app}}$) and the y value the $\mathrm{scr}^{\min}_{\leftrightarrow}$ ($\mathrm{scr}^{\min}_{\updownarrow}$) score achieved by another rule. If a point is on the blue line, the scores reached by the two rules are the same.

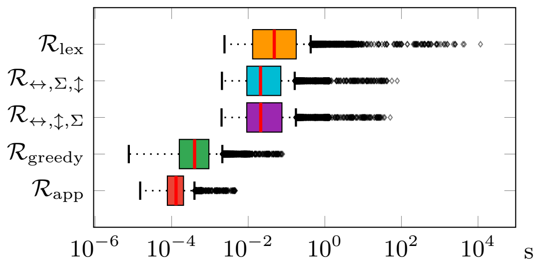

Figure 8. The runtimes of the rules for finding a winning committee sequence for the experimental data.

the right-hand scatter plot in Figure 7: On average, both $\mathcal{R}_{\mathrm{greedy}}$ and $\mathcal{R}_{\mathrm{lex}}$ reach only around 60% of the optimal $\mathrm{scr}^{\min}_{\updownarrow}$ score, which shows the costs of optimizing $\mathrm{scr}^{\min}_{\leftrightarrow}$ and ems.

5.4 *The Runtimes*

Figure 8 shows the rules' runtimes for finding one winning committee sequence. For each rule and instance, the runtimes of three separate computations on an Apple MacBook Pro (Apple M2 Max, 64 GB RAM, macOS Sonoma 14.4.1, Python 3.11) were averaged.

While all rules need less than 1 s for the vast majority of the experimental data, the results still reflect the fact that determining a winning committee sequence with $\mathcal{R}_{\mathrm{lex}}$, $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$, or $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ is NP-hard, while it is in P for $\mathcal{R}_{\mathrm{app}}$ and $\mathcal{R}_{\mathrm{greedy}}$. The rules $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ need at most around 51 s and 78 s, respectively, while $\mathcal{R}_{\mathrm{lex}}$ has the worst runtimes and needs around 3.2 h in the worst case ($n = 993$, $\tau = 40$, $m = 136$). This demonstrates the importance of a good heuristic solution.

5.5 *Investigating the Properties*

Even if a rule violates a given property in general, the conditions of that property can be satisfied by the rule for a specific instance. We therefore investigate how often our rules satisfy the conditions of a given property for the experimental data.

To avoid excessive runtimes, for each property we discard a specific instance, its variants, or a pair of two instances in any of the following two cases: (1) one of the rules that violate the property determines at least 30 winning committee sequences or (2) determining at most 30 winning committee sequences of $\mathcal{R}_{\mathrm{greedy}}$ takes more than 10 s using the C++ program and hardware mentioned in Sections 5.2 and 5.4, respectively. Under this restriction, relatively few instances are discarded and some properties are even tested on all our instances. Note that for some other properties we have further particularities or restrictions (e.g., upper bounds on n and m) to reduce runtimes.

Table 3. The (naturally rounded) proportion of the investigated (pairs of) instances for which a rule satisfies the conditions of the given property. If a rule satisfies a property in general, the corresponding cell contains a ✓.

Property	$\mathcal{R}_{\mathrm{lex}}$	$\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$	$\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$	$\mathcal{R}_{\mathrm{greedy}}$	$\mathcal{R}_{\mathrm{app}}$
P1	✓	✓	99.9	91.8	✓
P2	✓	✓	✓	98.6	84.7
P3	✓	✓	✓	98.0	✓
P4	✓	11.9	11.5	✓	✓
P5	✓	98.6	98.4	98.6	✓

Table 3 gives an overview of the results, in which the percentages are based on all instances or combinations thereof that are not discarded. For *Independent Groups (P4)*, the table only displays the results based on instances with at least two independent groups, for *Sub-Consistency (P5)*, we only consider instance pairs with at least one common winning committee sequence.

Our results show that $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ violates the condition of *$\leftrightarrow$-Pareto efficiency (P1)* very rarely for the experimental data. On the other hand, both $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ violate the conditions of *Independent Groups (P4)* for around 88% of the instances with more than one independent group, so that $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ and $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ do not behave as one would expect from an egalitarian rule in such situations.

$\mathcal{R}_{\mathrm{greedy}}$ satisfies the conditions of *United h-Superadditive (P2)*—which is the only central property violated by $\mathcal{R}_{\mathrm{app}}$—for 98.6% of the instances, in contrast to 84.7% in the case of $\mathcal{R}_{\mathrm{app}}$. In addition, for each property, $\mathcal{R}_{\mathrm{greedy}}$ satisfies the corresponding condition for at least 91.8% of the instances, which fits the observation from Section 5.3 that $\mathcal{R}_{\mathrm{greedy}}$ seems to be a good heuristic for $\mathcal{R}_{\mathrm{lex}}$.

6 Epilogue

We introduced and adapted new properties and rules for (computing) egalitarian committee sequences as well as tested them against each other, both theoretically and experimentally. Our work promotes $\mathcal{R}_{\mathrm{lex}}$ for egalitarian committee sequences. While computationally demanding, $\mathcal{R}_{\mathrm{lex}}$ fulfills many desirable properties. Many of those are violated by $\mathcal{R}_{\mathrm{greedy}}$, which, however, performs well in our experiments—both regarding runtime and solution quality. Thus, $\mathcal{R}_{\mathrm{greedy}}$ qualifies itself as a good heuristic for $\mathcal{R}_{\mathrm{lex}}$, in particular on larger instances or when (computing) time is limited. If one accepts a smaller ems score for a higher sum of scores or a higher minimum vertical score, then $\mathcal{R}_{\leftrightarrow,\Sigma,\updownarrow}$ and $\mathcal{R}_{\leftrightarrow,\updownarrow,\Sigma}$ can be good choices.

Our work constitutes the first axiomatic and experimental study on egalitarian committee sequences and hence paves the way for future work in many directions. Interesting further properties to formulate or adapt include those connected to "strategyproofness" (cf. [15]). Future work can add further properties and rules not only for committee sequences that are egalitarian, but also for, e.g., equitable committee sequences (where every two agents score equally). Characterizations of our rules (as we have as preliminary result for $\mathcal{R}_{\mathrm{lex}}$) and impossibility statements are also worth to study. Regarding data and experiments, since most of our data is rather artificial, more real-world data is to collect. Also comparing winners between non-sequential elections and their "sequentialized" ones may be of interest.

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