Proportional Participatory Budgeting with Projects Interaction

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Abstract. Participatory budgeting (PB) is a democratic process for allocating funds to projects based on the votes of members of the community. PB outcomes are commonly evaluated for how they reflect voters' preferences (e.g., social welfare), and the extent to which they are fair (e.g., proportionality). Due to practical and computational reasons, voters are usually asked to report their preferences over projects separately, possibly neglecting important dependencies among projects which cause the outcome no longer being proportional, in addition to achieving lower satisfaction.

This work is the first to suggest a polynomial-time aggregation method capable of guaranteeing proportional outcomes under substitution dependencies. The method is based on the Method of Equal Shares, and we further provide an FPT variation that can guarantee a more relaxed notion of proportionality for any type of dependency. Through simulations, we demonstrate that these aggregation methods achieve, on average, at least as much social welfare as their counterparts that ignore the dependencies.

1 Introduction

Participatory budgeting (PB) is gaining increased attention from both researchers and practitioners and is actively in use in cities around the world [24, 1]. This process typically includes several steps. In the first step, citizens suggest and discuss different projects, followed by a stage of shortlisting [23] to get a short list of feasible projects with their cost estimations. Next, citizens vote on which of the projects they would like to be funded. Different input formats exist that allow voters to express their preferences [4, 11] over projects, such as approval voting, knapsack ranking and additive utility (specified later on). Finally, the votes are aggregated by a mechanism that selects a subset of projects to fund [25, 19, 21].

In assessing aggregation methods, various criteria are employed, among which the notion of proportionality. Proportionality aims to ensure that a group of voters with similar preferences receives a satisfactory level of welfare. One prominent polynomial-time aggregation method, the Method of Equal Shares (ES) [21], provides assurance that its outcome will satisfy the proportionality notion of *Extended Justified Representation up to 1 project* (EJR-1), under the assumption that voters' utilities over projects are additive.

In the real world, voters' preferences may exhibit complex relationships between projects. For example, the utility from one project might depend on whether some other project is funded or not [17, 18, 3, 15]. Consider for example a city where there are 4 different suggestions to build a large parking lot in different places, as well as two unrelated projects (say, a playground and a library). The budget is sufficient for only four projects. There is a severe parking problem, so most citizens will assign higher utility to parking lots than to the other projects (or rank parking lots higher than other projects). As a result all 4 parking lots are likely to be funded and consume the entire budget, even if one or two parking lots are sufficient to solve most of the parking shortage, and the remaining budget would be put to a better use by funding the other projects. This problem occurs since common mechanisms ignore the fact that the four parking lots are *substitutes*.

This example directly shows how ignoring such dependencies may lead to lower welfare. Furthermore, outcomes that satisfy EJR-1 are no longer guaranteed to exist when taking into account interactions. We will explain this in details in Section 4.2.

Our contribution.

- We provide two extensions of ES: *Interaction Equal Shares* (IES) that chooses the next project according to its marginal utility; and *Partition IES* (PIES) that considers subsets of interacting projects at each iteration.
- We show that even with few substitute projects, it is no longer guaranteed that an outcome satisfying EJR-1 exists. Therefore, we extend the notion of EJR-1 to consider arbitrary interactions between projects, namely *EJR with Interactions up to 1 project* (EJRI-1).
- We prove that IES always returns an EJRI-1 outcome under substitutes relation between projects; and that PIES holds a more relaxed notion of proportionality for any type of interactions.
- We show through simulations that IES and PIES achieve at least as high welfare on average as ES.
- Our simulations are performed by extending the open source library pabutools [16] to support aggregation with interaction between projects. This code will be made public for future research.

1.1 Related Work

Evaluation of aggregation methods. Past work in PB use different methods in order to evaluate the voting rules. One method for evaluation which received a lot of attention is proportionality, having many notions which are surveyed by Rey and Maly [22]. In our work, we will focus on the notion of Extended Justified Representation (EJR) which will be defined in Section 4.

Additional popular methods for evaluation include social welfare to evaluate mechanisms, such as Goel et al. [14], which shows that when using the knapsack input format (1/0 additive utilities, i.e., approval), it is possible to achieve an outcome that maximizes social welfare. Jain et al. [17] consider special cases where it is possible to find a polynomial time algorithm that maximizes the social welfare given interactions between projects.

There is a wide range in the literature on PB that talks about the tradeoff between welfare and proportionality. Fairstein et al. [10] studied the tradeoff between welfare and representation while checking how requiring proportionality affects them, showing that this requirement can lower the maximal achievable welfare. In addition, Michorzewski et al. [20] tests the relation between fairness (proportionality) and welfare, however, in the settings where projects are divisible and not required to be funded entirely. Stronger positive results can be seen for projects with unit cost [6].

Projects Interaction in PB. The literature suggests many different ways to represent voter preferences over a discrete set of projects (i.e., input format), such as approval voting [14, 12], knapsack voting [14, 13], ranking [2, 4], reporting utilities [21] for each of the projects and more. However, all of those methods ignore interactions among projects (such as substitution). There are several works which tackle this issue: Jain et al. [17, 18] describe an interaction structure based on a combination of project partitioning and approval voting. This is followed by complexity analysis for which cases it is possible to find the optimal outcome in terms of welfare. We will adopt this model for our paper.

As we described, there is a variety in the literature that consider the settings where there are project interactions. However, there is a lack of literature that consider proportionality in these settings. The only positive result we are aware of is regarding Fully Justified Representation (FJR). While their paper focuses on additive utilities, Peters et al. [21] mention that the definition of FJR allows for arbitrary interactions, and that the Greedy Cohesive Rule (which has exponential runtime) is guaranteed to find an FJR outcome (in particular one must exist).

Goyal et al. [15] also assume some partition over the projects with specific types of interactions. For this structure they suggested a voting rule which they show that under some constraints is strategyproof and finish with a complexity analysis for finding the optimal outcome in terms of welfare under their model. In comparison, Baumeister et al. [3] suggested model have two main differences, first, they work in multi-winner settings and second, they do not assume a partition over the projects; instead, each voter has their own partitions. They focus on a specific structure of project interaction (which can be considered as a special case of the work of Jain et al. [17]) and perform an axiomatic analysis on different voting rules for this model. Finally, Durand and Pascual [8] Considers project synergies during aggregation, highlighting important properties. However, finding the optimal outcome is NP-hard, and even approximations limiting synergy size can be impractically slow as size increases

Interactions Outside of PB. It worth mentioning that the subject of non-additive, or 'combinatorial' utilities spread much further in the literature beyond participatory budgeting. The challenges that we described in the introduction also exists in other areas of economics. For example, in combinatorial auctions under different utility functions Blumrosen and Nisan [5], and in fair allocation, where Chaudhury et al. [7] study the problem of allocating indivisible goods under subadditive valuations for the items. Decreasing marginal gain is also a common assumption in *continuous* budgeting problems [9, 26].

2 Preliminaries

The model is based on the project interaction model described by Jain et al. [17]. We start by defining the Participatory budgeting instance as an election tuple $(P, cost, Z, L, V, F)$, where:

- $P = \{p_1, \ldots, p_m\}$ is a set of m projects which are considered.
- cost: $P \to \mathbb{R}_+$ is a function specifying the cost of each project $p \in P$. The cost for subset of alternatives $T \subseteq P$ is $cost(T) = \sum_{p \in T} cost(p)$.
- $L \in \mathbb{R}_+$ is the total budget the voters have in order to fund the selected projects.
- A partition $Z = \{z_1, \ldots, z_{|Z|}\}$ of interaction sets of projects, where $\cup_{i \in [|Z|]} z_i = P$ and $\forall i, j \in [|Z|]$; $z_i \cap z_j = \varnothing$.

The preferences elicited from voters are composed of two parts:

- Each of the n voters $V = \{v_1, \ldots, v_n\}$ submits an approval ballot, where v_i specifies the set of projects $T \subseteq P$ approved by voter i.
- To convey the utility they derive from the approved projects, voters define a set of interaction functions $F_i = \{f_{iz_1}, \ldots, f_{iz_{|Z|}}\}$, where $\forall z \in Z$; $f_{iz} : \mathbb{Z}^+ \to \mathbb{R}^+$.

We mark by $f_{iz}(k)$ the utility that voter i gets from an interaction set z in which there are exactly k projects approved by the voter that were chosen in the outcome W. Formally, the utility that voter i gets from outcome W is: $u_i(W) := \sum_{z \in Z} f_{iz}(|z \cap v_i \cap W|)$, and the *marginal utility* that a voter i gets from adding p to subset of projects $T \subset P$ is denoted by $u_i(p|T) := u_i(T \cup \{p\}) - u_i(T)$.

Throughout the paper, we will consider only interaction functions that satisfy the following:

1. $f(0) = 0$: no utility is received if the voter does not approve any project from a part $z \in Z$ that is funded.
2. $\forall k > k', f(k) \geq f(k')$: the interaction function is non-decreasing.

In addition to the general model, we will look at a family of interaction functions that describe substitution between projects in the same partition, [1] i.e. $f(k+1) - f(k) \leq f(k) - f(k-1)$.

Note that the way that the interaction function is defined, it depends only on how many projects were funded from the same group and is indifferent about which project is funded.

Finally, given a PB instance, an *aggregation method* will return a feasible outcome, i.e. $W \subseteq P$ with $cost(W) \leq L$.

If we go back to the Example from Section 1, we have a partition where all parking lots are in the same set z_1 and the other projects are each in a separate set. A voter that wants at most 2 parking lots might express his interaction function as $f_{iz_1}(k) = (1, 2, 2, 2)$ i.e. f equal to 1 for $k = 1$ and 2 for $k \in [2, 4]$. If one wants to simplify the voting process, they can provide default interaction functions, allowing the voters the option to override it, or only submit an approval ballot.

3 Aggregation Algorithm

Our starting point is the Method of Equal Shares (ES) algorithm introduced by Peters et al. [21]. ES is an iterative rule, which starts with

[1] We do note that some rules, such as some Thiele methods, can be thought of as maximizing welfare under the assumption that *all* projects are substitutes to some extent.

"allocating" each voter an equal share of the budget $\frac{L}{|V|}$, initializes an empty outcome $W = \varnothing$; then it sequentially adds projects to W. At each step, to choose some project $p \in P \setminus W$, each voter needs to pay an amount that is proportional to her utility from the project, but no more than her remaining budget (note that with approval utilities this means only agents that approve the project pay). The total payment should cover the cost of the project.

Formally, let $b_i(t)$ be the amount of money that voter i is left with just before iteration t. We say that some project $p \in P$, is q-affordable if $\exists q \in \mathbb{R}_+$ such that

$$\sum_{i \in V} min(b_i(t), u_i(p) \cdot q) \geq cost(p) \tag{1}$$

Where $u_i(p)$ is the utility of voter i for project p. If project p is q-affordable, we will denote by $pay_i(p) := min(b_i(t), u_i(p) \cdot q)$, how much voter i needs to pay for project p if it is funded.

If no candidate project is q-affordable for any q, ES terminates and returns W. Otherwise it selects project $p^{(t)} \notin W$ that is q-affordable for a minimum q, where individual payments are given by $pay_i(p^{(t)})$. We then update the remaining budget to $b_i(t+1) := b_i(t) - pay_i(p^{(t)})$.

We emphasize that ES assumes additive utilities and does not take into account project interactions (recall the parking lots example from the introduction).

Aggregation with Interactions. We will now extend ES to *Interaction Equal Shares* (IES). During the run of IES, there are no fixed project utilities, but rather *marginal utilities* that are updated with every iteration: in each iteration where the set of projects $B \subseteq P$ was chosen so far, we calculate the qValues (see Eq. (1)) using $u_i(p|B)$ instead of $u_i(p)$. Note that both ES and IES run in polynomial time.

The following example shows how each of the mechanisms work on an instance with substitutes.

Example 1. *Consider the PB scenario where $V = \{v_1, v_2\}$, $P = \{a, b, c, d, e\}$. In the following table, the first line describes the interaction functions for each interaction set (both voters have the same functions), followed by the approval sets of each voter.*

	$\{a\}$	$\{b, e\}$	$\{c, d\}$
f	(1)	$(1, 1.2)$	$(1, 1.2)$
v_1	$\{a\}$	$\{b\}$	$\{c, d\}$
v_2	$\{a\}$	$\{b, e\}$	$\{c\}$

The budget $L = 2$ and $cost(a) = \frac{11}{10}, cost(b) = cost(c) = 1, cost(d) = cost(e) = \frac{1}{3}$. In the case that ES is used, it has no knowledge of interactions, so all projects just have utility of 1.

ES *Project a is $\frac{11}{20}$-affordable; projects b and c are $\frac{1}{2}$-affordable; projects d and e are $\frac{1}{3}$-affordable. ES takes the project with lowest qValue (using lexicographic tie-breaking), which is d, followed by e as values do not change over iterations.*
Each voter is left with a budget of $\frac{2}{3}$, and projects are still $\frac{11}{20}$-affordable for a and $\frac{1}{2}$-affordable for b and c. In the next iteration, project b will be chosen and ES will terminate as there are no q-affordable projects. The final bundle of chosen projects is $W_{ES} = \{b, d, e\}$ and the social welfare is $1 + 1 + 1 + \frac{1}{5} = 3\frac{1}{5}$ as voter 1 got two substitute projects.

IES *This procedure begins the same way as ES as no project was funded yet and all utilities are 1, therefore projects d and e are funded in the first two iterations.*
Since project b is substitute for v_2 and project c is substitute for v_1, their utility changes to $u_2(b) = u_1(c) = \frac{1}{5}$ in the next iteration, making them $\frac{5}{6}$-affordable. Since project a is not affected it stays $\frac{11}{20}$-affordable.
Project a has the lowest qValue, therefore, it will be chosen and IES will terminate as no item is q-affordable anymore. The outcome is $W_{IES} = \{a, d, e\}$ and the social welfare is $1+1+1+1 = 4$ (project a provides utility 1 for each voter, project d provides utility 1 for v_1 and project e provides utility 1 to v_2).

As can be seen from the example, when using ES, voter 1 gets two substitute projects, while when using IES, the mechanism will prioritize using voter funds for projects that are not substitutes even if they are more costly. A more detailed welfare analysis of the different methods can be found in Section 5.

4 Proportionality

We start this section by mentioning the notion of Fully Justified Representation [21] (FJR), which was defined for additive utilities but naturally extends to utilities with arbitrary interactions. Peters et al. [21] suggests the Greedy Cohesive Rule (GCR) which satisfies FJR (regardless of interactions), but runs in exponential time. In addition, they show that FJR implies a weaker notion of proportionality called Extended Justified Representation (EJR-1) which will be defined below.

In the rest of this section, we revisit the definition of EJR-1 and the proof of ES holding it in the additive setting. Then, we show that in interaction settings, there may not always exist an outcome that satisfies EJR-1. As a solution, we give an extension of EJR-1 suited for such settings, revealing ES may not meet this criterion. Finally, we demonstrate that IES satisfies this extension with substitute interactions and propose a variation of IES that satisfies a relaxed proportionality notion for all interaction types.

4.1 Proportionality in the Additive Setting

Let us recall the proportionality definition for additive utilities [21] where each voter i gets a utility of $u_i(p)$ for project $p \in P$. For a function $\alpha : P \to [0, 1]$, we mark $\forall T \subseteq P; \alpha(T) := \sum_{p \in T} \alpha(p)$.

Definition 1 (Cohesive group [21])**.** *A group of voters S is (α, T)-cohesive, where $\alpha : P \to [0, 1]$ and $T \subseteq P$, if $|S|/n \geq cost(T)/L$ and if $u_i(p) \geq \alpha(p)$ for all $i \in S$ and $p \in T$.*

Definition 2. *(Extended Justified Representation Up To One Project—EJR-1 [21]). A rule R satisfies EJR-1 if for each election instance E, each $\alpha : P \to [0, 1]$, $T \subseteq P$, and each (α, T)-cohesive group of voters S there exists voter $i \in S$ such that either $u_i(R(E)) \geq \alpha(T)$ or for some $p^* \in T$ it holds $u_i(R(E) \cup \{p^*\}) > \alpha(T)$.*

We next decompose the main parts of the proportionality proof from [21], in order to see which parts should be amended once interactions are introduced. The proofs of the presented propositions can also be found in Appendix A for completeness.

Consider some PB instance and a (α, T)-cohesive group S. Denote $\alpha := \sum_{p \in T} \alpha(p)$. We must show there is some $i \in S$ with utility at least α, or exceed it with one additional project.

The proof starts by defining three separate runs:

(A) ES runs on the original instance and outputs the outcome W.
(B) ES runs on the original instance, but voters in S have no budget constraints when paying for candidates in T.

(C) ES runs with only voters S and projects in T, unlimited budgets, and where $u_i(p) = \alpha(p)$ for all voters and projects.

Note that if at the end of run (B) no voter in S exceeds their initial funds of $\frac{L}{n}$, then the outcome of (B) is the same of (A) and all projects in T are selected so we are done.

In the case that some voter i^* pays more than $\frac{L}{n}$ in run B, we consider the iteration voter i^* first exceeds her budget, and p^* is selected. Let W' be the set of projects selected before p^* (which are the same in run A and B).

Definition 3. *(cost-utility function). The function $f_{(B)}(x)$ represents the amount of funds that voter i^* had to spend during the run (B) until receiving a utility of x.[2] The function $f_{(C)}(x)$ is similarly defined for run (C).*

The following three propositions are not stated as such in [21], but are shown there as part of the main proof:

Proposition 1 ([21])**.** *$f_{(C)}(x)$ is a convex function.*

Note that in run (C) as there are unlimited funds, all projects are always σ_p-affordable [3] and thus it holds for any project $p \in T$:

$$\sum_{i \in S} \sigma_p \alpha(p) = |S| \sigma_p \alpha(p) = cost(p) \tag{2}$$

Proposition 2 ([21])**.** *Under run (B) at each step, any not-yet-selected $p \in T$ is ρ-affordable for some $\rho \leq \sigma_p$.*

Proposition 3 ([21])**.** *$f_{(B)}(x) \leq f_{(C)}(x)$ for all $x \in [0, \alpha]$.*

Given the point α it holds:

$$f_{(C)}(\alpha) = \sum_{p \in T} \sigma_p \cdot \alpha(p) = \sum_{p \in T} \frac{cost(p)}{|S|} = \frac{cost(T)}{|S|} \leq \frac{L}{n}$$

And by Proposition 3 it holds:

$$f_{(B)}(\alpha) \leq \frac{L}{n} \tag{3}$$

To conclude, consider the state exactly after (B) adds project p^*. Voter i^* hast spent at least $\frac{L}{n}$ at this point. There are two cases:

1. i^* spent exactly $\frac{L}{n}$. In this case p^* is also selected by (A) as (A) and (B) behave the same until this point. According to Eq. (3) $u_{i^*}(W_{(B)} \cup p^*) \geq \alpha$, thus $u_{i^*}(W) \geq \alpha$.
2. i^* spent strictly more than $\frac{L}{n}$. In this case, by definition of (B), we have $p^* \in T$ such that Eq. (3) implies $u_{i^*}(W_{(B)} \cup p^*) > \alpha$. As $W_{(B)} \subseteq W$, it implies $u_{i^*}(W \cup p^*) \geq \alpha$.

In both cases, W satisfies EJR-1.

4.2 *Proportionality In The Projects Interaction Settings*

Recall that some notions of proportionality are guaranteed to exist even with arbitrary interactions, e.g. FJR [21]. Yet, while FJR implies EJR in the additive case, this is no longer true once interactions are introduced, and in fact EJR-1 may not hold for any outcome.

To see this consider the simple instance with a single voter that approves two substitute projects $P = z = \{p_1, p_2\}$ with enough budget to fund both and $f_z(k) = (1, 2)$. It is easy to see that the single voter is cohesive for $T = P$ and $\alpha \equiv 1$, so $\alpha(z) = 2$, therefore for W to hold EJR-1 we require that either $u_i(W) \geq \alpha(z)$ or there is some project $p^* \in z$ such that $u_i(W \cup \{p^*\}) > \alpha(z)$. However, even funding both projects (i.e. $W = P$) only provides a total utility of 1—strictly lower than the required utility of $\alpha(z) = 2$.

This demonstrates an issue where an outcome W can achieve lower utility than the utility we guarantee due to W because α is additive while the utility is not. Even if we explicitly disallow a set from blocking itself, there are other instances where there is no outcome that holds EJR-1. For a more detailed example of such a scenario see Appendix B.

As can be seen in the example, the reason that there might not be an outcome that satisfies EJR-1 is related to Def. 1 where α is only defined for singletons. Therefore, we would like to extend the cohesiveness definition such that α will also consider sets of projects. For $p \in P, B \subseteq P$, let $MU_\alpha(p, B) := \alpha(B \cup p) - \alpha(p)$. Difference from Def. 1 is highlighted.

Definition 4 ((α, T)-cohesive with interactions)**.** *A group $S \subseteq V$ of voters is (α, T)-cohesive for a set of projects $T \subseteq P$ and $\alpha : P^{|P|} \to R^+$, if $|S|/n \geq cost(T)/L$ and for any project $p \in T$ and a subset of projects $B \subseteq P \setminus \{p\}$ it holds $u_i(p|B) \geq MU_\alpha(p, B)$.*

Extended Justified Representation with interactions up to one project (EJRI-1) is the same as Def. 2, except we now use Def. 4 for cohesive groups.

Under the new definition, we again always have at least one outcome that satisfies EJRI-1 (See next section that IES outcomes always satisfy EJRI-1). For example, going back to the example above, the single voter is still cohesive for the two projects, however under Def. 4, $\alpha(P) = 1 = u_i(p_1)$. Thus any outcome with at least one project holds EJRI-1.

We argue that EJRI-1 is the correct extension for the interaction settings for three reasons: first, EJRI-1 collapses to the EJR-1 definition when the projects utilities are additive. Second, FJR implies EJRI-1 with arbitrary interactions (see Prop. 5 in the appendix). Finally, as the next section shows, IES holds EJRI-1 under substitute interactions (and runs in polynomial time).

To conclude this section, we note that EJR-1 and EJRI-1 are incomparable i.e. there are instances where outcome that satisfies EJR-1 does not satisfy EJRI-1 and the other way around. One exception is when we have only substitute interactions, then EJR-1 implies EJRI-1. See Prop. 4 in Appendix A and discussion in Appendix B for details.

4.3 *Proportionality of IES and PIES*

Next, we will show that IES holds EJRI-1 if all interaction functions are for substitute projects as defined in Section 2. The proof will be an adaptation of the proof shown in the previous section for additive utilities. The following changes are required:

1. Defining (C) for setting with projects interaction.
2. Proving that Proposition 3 holds for this settings.

Given those changes, the rest of the proof stays the same.

We start with defining (C): IES runs on a smaller instance with only voters S and projects in T, with an unlimited budget. The interaction functions are defined such that the following holds:

$$\forall T' \subseteq T, i \in S; u_i(T') = \alpha(T')$$

[2] To make the function continuous each two points are connected by a straight line such that the utility of a project is given proportional to how much of its cost has been paid so far.

[3] σ_p is the projects qValue

Proof of Proposition 1 under substitutes. We note that as we consider only substitution interaction functions, the marginal utility of each project can only decrease between interactions while their cost left unchanged. For this reason, their ratio between cost to utility i.e. σ_p can only increase. Since IES always takes the project with the lowest ratio and the ratio of the other project can only increase between iterations, the slope of $f_{(C)}(x)$ can only increase. □

This proposition means that the ratio of projects selected in (C) can only be higher as it progresses.

Next, we note that Proposition 2 does not hold in our settings. The reason is that a project that will be selected in (C) first from its interaction set, can be chosen later in (B) lowering its utility and thus the same project has lower ratio in (B) compared to (C).

Therefore, we will skip this proposition, and prove directly Proposition 3, i.e. that $f_{(B)}(x) \leq f_{(C)}(x)$:

Proof of Prop. 3 under substitutes. Given some point $x' \in [0, \alpha]$ which isn't a boundary point for either $f_{(C)}(x)$ or $f_{(B)}(x)$, we say that x' is on the segment that correlates to some project $d \in P$ on $f_{(B)}(x)$ (call it segment 1) and on segment that correlates to some project $p_s \in T$ on $f_{(C)}(x)$ (call it segment 2). See Figure 1 for an illustration. Consider the time point t where (B) chooses to add d (before adding it). At time t, i^* utility in (B) equal to the x-coordinate of the (B) segment, thus less than x'.

We note by $B_{x'}$ and $C_{x'}$ the projects selected by (B) and (C) until getting to utility x' (without p_s and d). There is some interaction set $z \cap T \neq \varnothing$ such that $z_B := |B_{x'} \cap z| \leq z_C := |C_{x'} \cap z|$. If this is not the case, it means that $C_{x'} \subset B_{x'}$, however it not possible that at point x' (B) have selected all projects as (C) and more as it will have higher utility.

Therefore, there is some project $p_{j'} \in z$ such that $j' := max_k\ k \leq s$. In addition, as $z_B \leq z_C$, there is some project $p_j \in z$ with $j \leq j'$ such that p_j was not selected at time t by (B). At this time, the utility for this project in (B) is at most as the utility of $p_{j'}$ when selected in (C) as there is at most the same amount of substitutes from z at time t in (B). In addition, (C) chooses projects from the same interaction set from cheapest to most expensive (as they share the same utility in each iteration), thus $cost(p_j) \leq cost(p_{j'})$.

As a result, the cost-utility ratio of p_j in (B) is at most as the ratio of $p_{j'}$ in (C). Furthermore, from Proposition 1, the ratio of $p_{j'}$ in (C) is at most the ratio of p_s in (C). Since (B) always chooses projects that have the smallest ratio it must be that d has a ratio at most as this of p_j in (B). Accordingly, this means that d also has a ratio at most as the ratio of p_s in (C).

Since (B) always chooses a project that is ρ-affordable with the smallest ρ it must be that d is ρ-affordable with $\rho \leq \rho'$. This mean that the slope at (B) segment is at most ρ and therefore at most σ_{p_j}. On the other hand, the segment on (C) have slope of p_s which is weakly lower than p_j. Therefore, (B) segment have weakly lower slope than (C). As it is true for any x' (except boundary points) the proposition holds. □

While IES holds EJRI-1 when using substitutes, this is not true anymore for the general case, which can also be seen in the following example:

Example 2. *Given PB scenario with 3 voters $\{v_1, v_2, v_3\}$ and 5 projects $\{p_1, p_2, p_3, p_4, p_5\}$ where $cost(p_3) = cost(p_4) = cost(p_5) = 1, cost(p_1) = cost(p_2) = 1.5$, the budget $L = 3$ and the voters preference shown in the following table:*

	$\{p_1, p_2\}$	$\{p_3\}$	$\{p_4\}$	$\{p_5\}$
f	$(0, 10)$	(1)	(1)	(1)
v_1	$\{p_1, p_2\}$	$\{p_3\}$	$\varnothing$	$\varnothing$
v_2	$\{p_1, p_2\}$	$\varnothing$	$\{p_4\}$	$\varnothing$
v_3	$\{p_1, p_2\}$	$\varnothing$	$\varnothing$	$\{p_5\}$

Since the utility for the first project from p_1 and p_2 is zero, neither of those project will be ever selected by IES. Instead, projects $\{p_3, p_4, p_5\}$ will be selected which results in utility 1 per voter.

Note that all voters are cohesive over the project set $\{p_1, p_2\}$ and should guarantee a utility of 10 to at least one voter. Adding another project will add the utility of zero, which would not guarantee EJRI-1 either.

To support proportionality for general interactions IES can be modified to consider all subsets of each $z \in Z$ at each iteration, we will call this method Partition IES (PIES) and it is defined more formally in appendix C. This has two drawbacks: one is that runtime is exponential in $|z^*| := \max_{z \in Z} |z|$; and the other is that the obtained proportionality notion is further relaxed to EJRI-z instead of EJRI-1, i.e. instead of requiring EJRI up to one project $p \in T$, we require EJRI up to one interaction set $z \in Z;\ z \cap T \neq \varnothing$ (see appendix C for proportionality proof).

As can be seen, PIES guarantee EJRI-z which is weaker compared to EJRI-1, promising a cohesive group a lower welfare guarantee. However, in the case that all interactions sets are relatively small the guarantees is not much weaker compared to EJRI-1.

Given the results so far, one might ask, what happen if we relax the model such that instead of single partition Z, each voter defines its own partition. Regrettably, IES does not hold EJR-1 or EJR-z in this scenario. More details about it can be found in the appendix D.

5 Welfare Analysis

So far we saw under what conditions IES and PIES are proportional, however one of the main purposes of extending the way to express preferences is to allow for higher social welfare. For this reason, in this section we will focus on analysing the social welfare that can be achieved by the different methods.

5.1 Worst-Case Analysis

We remind the reader that in the approval settings (without interactions) it was shown that the welfare ratio between ES and the optimal welfare is bounded by the number of voters, and bounds remain similar for all proportional rules [10]. In contrast, when moving to additive utilities this ratio can be much worse. This is because ES can choose project with the lowest utility while the optimal outcome will take the projects with the highest utility. Since this can happen for all voters between each pair of projects, we get that it is possible for ES to get a welfare ratio as bad as $O(|P| \cdot |V| \cdot \frac{\text{max utility}}{\text{min utility}})$.

Since utilities with interactions only generalize additive utilities further, they are susceptible to similar issues. However, it is not possible to get worse welfare ratio then the one described (i.e. the bound is tight), as every voter can have at most $O(|P|)$ projects which it gets with the minimum utility instead of the maximal utility. For this reason, the worst-case welfare approximation bounds of ES and IES are equally bad. See Appendix E for details.

However, for specific instances, one rule might be better than others. Next, we would like to answer whether one of the aggregation methods ES / IES / PIES is superior to the other in terms of welfare. To answer this we start with the following example:

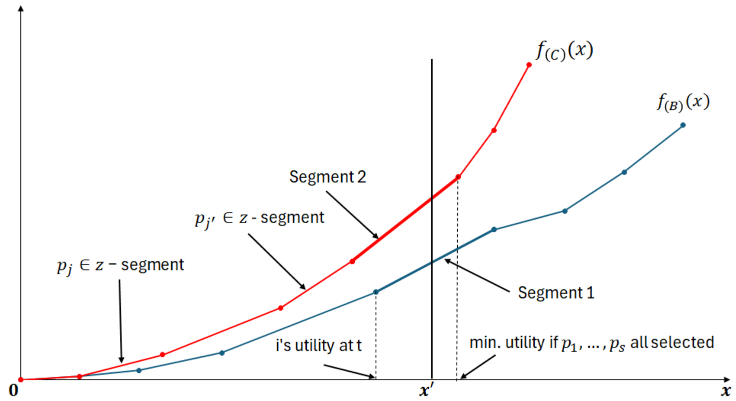

Figure 1: Illustration of the proof of Proposition 3. Based on Peters et al. [21, Figure 4]

Given partition $Z = \{z_1, z_2\}$ such that each includes half of the projects, the interaction functions for all voters are same:

$$f_{iz_1}(k) = \sum_{j=1}^{k} \begin{cases} 1, & \text{if } j = 1. \\ 5, & \text{otherwise.} \end{cases} \quad ; \quad f_{iz_2}(k) = \sum_{j=1}^{k} 5^{j-1} \tag{4}$$

For some small $\epsilon > 0$, we define the cost of one project in z_1 to be $1-\epsilon$ and the cost for the rest of the projects in z_1 to $1+\epsilon$. All projects in z_2 have cost of 1. The budget for this instance is $L = |z_1| = |z_2|$.

All projects "appear" to ES as with a utility of 1 so it will choose them by their price i.e. will take the cheap project from z_1 and $L-1$ projects from z_2 having the utility of $1 + f_{iz_2}(L-1)$. In contrast, we have IES which will also start with the cheap project from z_1 but after that the utility for the rest of the projects in z_1 updates to 5, causing IES to use the rest of the budget for only projects from z_1, which have total welfare of $f_{iz_1}(L-1)$. As can be seen, as the number of projects increases the ratio between ES and IES increases.

While this example show that ES welfare can be significantly higher that IES, by switching between z_1 and z_2 costs we can switch between their outcomes, thus having ES having significantly lower welfare.

Therefore, we see here that each of ES and IES can be much better or worse than the other for two almost identical instances.

PIES on the other hand choose the optimal outcome in both cases. However, there are scenarios where PIES can have significantly lower welfare compared to ES and IES. We will see this in the following example:

Example 3. *Given PB instance with three projects, one voter that approve all projects and a budget $L \geq 4$. The interaction functions are:*

	p_1, p_2	p_3
f	$1, 3$	L

$cost(p_1) = cost(p_2) = 1.1; cost(p_3) = L-1.$

When running PIES, we consider p_1/p_2 with 1.1-affordability, p_3 with $\frac{L-1}{L}-affordability$ and $\{p_1, p_2\}$ with $\frac{11}{15}$-affordability. Since $L \geq 4$, $\{p_1, p_2\}$ will be selected and PIES will stop. In contrast, both ES and IES which consider only one project at a time, will fund p_3, followed with one of the other projects and stop. This mean that PIES will stop with utility of 3 [4]*, while ES and IES stops with utility of $L+1$. Therefore, PIES welfare can be significantly low than the welfare of ES and IES.*

[4] We can also add additional project that cost $L-2.2$ with utility 1 so PIES will exhaust its budget

To conclude this section, we saw that in the project interaction settings neither of the methods ES, IES, and PIES dominate the other and there are instances where each of the methods can be better than the other.

5.2 *Average Case Analysis*

In the previous section we looked at the worst case of the aggregation methods, however, those cases usually happen in extreme cases which are not very likely to happen. Therefore, in this section, we would like to understand how they behave on average in a variety of instances.

Simulation Settings. For this purpose we run simulations with different types of interactions, comparing the average welfare the voters receive for four aggregation rules: ES (which does not take interactions into account), IES, PIES and proportional greedy (greedy by $utility/cost$). The reason for these rules is to see if taking interactions into account can help to achieve better welfare and having proportional greedy (PG) as a baseline, which is known to achieve good welfare.

The simulations include 50 projects with their cost sampled from Normal distribution $N(300, 30)$ and split randomly into partitions each of size $[1, 10]$. Next, 100 voters approve randomly 3 partitions each having one of six types of interactions:

1. Type 1 (Minimal substitutes) - the voter prefers one non-substitute project over the entire set of substitute projects i.e. the utility for the first project is 1, while any additional project has the utility of $1/m$.
2. Type 2 (harmonic) - the marginal utility decrease by harmonic series, formally: $f(i) = \sum_{j=1}^{i} \frac{1}{j}$.
3. Type 3 (exponential sub) - the marginal utility decreases exponentially, formally: $f(i) = \sum_{j=1}^{i} \frac{1}{1.5^{j-1}}$.
4. Type 4 (extreme complementary) - the voter significantly prefers at least two projects and wants as many as possible, formally the marginal utility for the first project is 1 and 50 for the rest.
5. Type 5 (linear) - the marginal utility increases linearly, formally: $f(i) = \sum_{j=1}^{i} j$.
6. Type 6 (exponential comp) - the marginal utility increases exponentially, formally: $f(i) = \sum_{j=1}^{i} 1.5^{j-1}$.

The list of interactions comprises three types of substitute projects (type 1-3) and three types of complementary projects (type 4-6), each with a welfare value of 1 for the first project in each interaction set. This deliberate choice is aimed at initially obfuscating which interaction set possesses the highest potential, thereby requiring IES to do some "exploration" which is possible as more projects can be funded. An example of such a scenario was presented in Section 5.1.

To perform the experiments Pabutools [16] library was used. As the library currently supports additive utilities but no interactions between projects, therefore we extended it to support this settings.

Simulation Results. The results in Figures 2- 3 are created by calculating the average voter welfare for each simulation as described above followed by averaging this value over 1000 simulations. The simulations included 9 settings: all voters have the same interaction type, each interaction function chosen randomly from types 1-3 or types 4-6 and each interaction function chosen randomly from types 1-6 (both substitute and complementary). Due to similarity in the results, we present here only 4 settings, where the rest can be found in the appendix F.

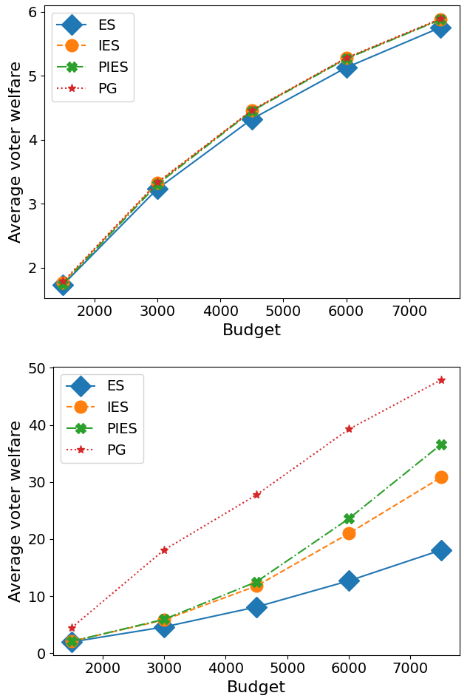

Figure 2: Average voter welfare over 1000 simulations, given only exponential sub voters (top) or only exponential comp voters (bottom).

Observing Figure 2, we note that with a limited budget and few projects fundable, the difference in welfare between ES and IES/PIES is minimal, as interactions exert minimal influence on utilities. However, as the budget increases, the welfare gap widens, indicating greater impact of interactions on project selection. Nonetheless, as nearly all projects become fundable, this gap is expected to narrow.

In the substitutes scenario, both IES and PIES achieve welfare levels nearly equivalent to PG, while in the complementary scenario, they outperform ES but fall short of PG. Despite all three (PG, IES, and PIES) prioritizing project selection based on cost-to-utility ratio, IES and PIES focus on ensuring fairness over maximizing welfare. Notably, in scenarios with uniform interaction types and thus agreement on project utility, PIES achieves higher welfare by considering sets of projects in each iteration.

Figure 3 top extends the initial observation with a mix of complementary interactions. Yet, due to the high variance in voter utilities for projects, PIES may prioritize interactions sets with lower welfare, resulting in lower welfare compared to IES.

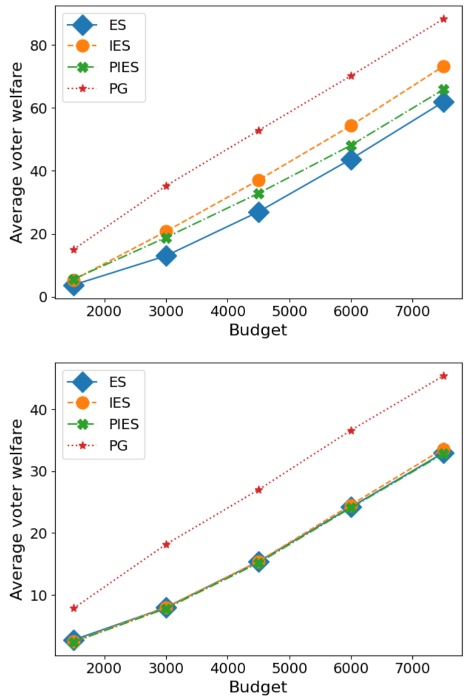

Figure 3: Average voter welfare over 1000 simulations, voters have mix of complementary interactions (top) or voters have mix of all possible interactions (bottom).

In Figure 3 bottom, where voters can utilize all interaction types, we observe the behavior outlined earlier. With a limited budget, IES/PIES lacks flexibility for exploration, potentially leading to suboptimal choices and slightly lower welfare compared to ES. However, as the budget expands, the gap diminishes until IES begins to surpass ES in welfare again (with PIES maintaining welfare close to ES).

6 Conclusion And Future Work

In this paper we considered proportionality in participatory budgeting in the setting where there is interaction between projects. We defined a variation of the EJR/EJR-1 proportionality axiom that takes interactions into account, and suggested two variations of the celebrated ES mechanism that are proportional under project interactions. The first (IES) runs in polynomial time and guarantees proportionality under substitutes; and the second (PIES) is approximately-proportional under arbitrary interactions, but its runtime may exponentially depend on the size of the interaction sets, and approximation guarantees similarly deteriorate.

We extended pabutools [16] to support aggregation with interaction between projects which was used to create synthetic simulations with different types of interactions. The simulations show that IES and PIES achieve better welfare compared to ES (which does not take interactions into account), except in extreme cases where there are both voters with substitute and complementary projects, and a low budget in addition.

A natural followup question is whether there is a polynomial time aggregation rule which holds EJRI-1 for general interactions. Furthermore, our work assumes there is a joint partition of the projects used by all voters, thus it raises the question whether there exists a polynomial time aggregation rule that guarantees proportionality while allowing voters to create their own partition for the substitute projects.

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A Omitted Proofs

Proof for Proposition 1 from Section 4.

Proof. For each $p \in T$, let us write $\sigma_p = cost(p)/(|S|\alpha(p))$. There is some order over the projects in T $\{p_1, \ldots, p_{|T|}\}$ such that $\sigma_{p_1} \leq \ldots \leq \sigma_{p_{|T|}}$. As ES choose projects by the ratio of cost to utility and there is unlimited funds, the projects will be selected by the described order and the slope of $Ff_{(C)}(x)$ will only increase. □

Proof for Proposition 2 from Section 4.

Proof. This holds as there are more voters in (B) compared to (C), thus projects in T have weakly stronger utilities. Formally:

$$\sum_{i\in S} \sigma_p \cdot u_i(p) + \sum_{i \in N\setminus S} min(b_i, \sigma_p \cdot u_i(p)) \geq$$
$$\sum_{i\in S} \sigma_p \cdot u_i(p) \geq \sum_{i\in S} \sigma_p \cdot \alpha(p) = cost(p)$$

□

Proof for Proposition 3 from Section 4.

Proof. Given some point $x' \in [0, \alpha]$ which isn't a boundary point for either $f_{(C)}(x)$ or $f_{(B)}(x)$, we say that x' is on the segment that correlates to some project $d \in P$ on $f_{(B)}(x)$ and on segment that correlates to some project $p_s \in T$ on $f_{(C)}(x)$. Consider the time point t where (B) choose to add d (before adding it). At time t, i^* utility under (B) equal to the x-coordinate of the (B) segment, thus less than x'. In addition, at this time it can't be the case that projects $p_1, \ldots, p_s$ are already selected in (B) as i^* utility will be at least $\alpha(p_1) + \ldots + \alpha(p_s)$ which is the right end of (C) segment which is more than x'.Therefore, there is some $p_j; j \leq s$ such that (B) did not selected yet p_j at time t. By Proposition 2, at time t p_j is ρ'-affordable in (B) for some $\rho' \leq \sigma_p$. Since (B) always choose project that is ρ-affordable with the smallest ρ it must be that d is ρ-affordable with $\rho \leq \rho'$. This mean that the slope at (B) segment is at most ρ and therefore at most σ_{p_j}. On the other hand the segment on (C) have slope of p_s which is weakly lower than p_j as $j \leq s$. Therefore, (B) segment have weakly lower slope than (C). As it is true for any x' (except boundary points) the proposition holds. □

In section 4 we mention the notion of EJR-1 and EJRI-1, which are a relaxation of EJR and EJRI. The difference between EJR and EJR-1 or EJRI and EJRI-1 is whether or not we allow proportionality "up to 1 project" i.e. we might need to add 1 project to satisfy it. Next, we will show that the stronger notions hold that EJR implies EJRI under substitute interactions and FJR implies EJRI for any interactions.

Proposition 4. *EJR implies EJRI under substitute interactions.*

Proof. Suppose that some aggregation method holds EJR and lets look at some (α, T)-cohesive group S. For every $i \in S$, every project $p \in T$ and a subset of projects $B \subseteq P \setminus \{p\}$ it holds $u_i(p|B) \geq MU_\alpha(p, B)$. We set an additive function $\alpha'(p) = \alpha(p)$ i.e. α' of a project equal to α of the same project without any subset of projects. In particular, for $B = \varnothing$ it holds $u_i(p) = u_i(p|\varnothing) \geq \alpha(p, \varnothing) = \alpha'(p)$. As the method holds EJR we have a voter $i \in S$ such that its outcome W holds $u_i(W) \geq \alpha'(T) \geq \alpha(T)$ as α can be only smaller when taking substitution into account. □

The proof that EJR-1 implies EJRI-1 under substitutes is similar.

Proposition 5. *FJR implies EJRI.*

Proof. Suppose that some aggregation method holds EJR and lets look at some (α, T)-cohesive group S. For every $i \in S$, every project $p \in T = \{t_1, \ldots, t_{|T|}\}$ and a subset of projects $B \subseteq P\setminus\{p\}$ it holds $u_i(p|B) \geq MU_\alpha(p, B)$. We set $\beta = \alpha(T)$. Then the following holds:

$$u_i(T) = \sum_{j=1}^{|T|} u_i(t_j | \cup_{k=1}^{j} t_k) \geq \sum_{j=1}^{|T|} MU_\alpha(t_j, \cup_{k=1}^{j} t_k) = \alpha(T)$$

Thus S is weakly (β, T)-cohesive. As the method hold FJR we have a voter $i \in S$ such that its outcome W holds $u_i(W) \geq \alpha(T)$, as required. □

B Omitted Examples

In section 4.2 we saw that there isn't always an outcome that satisfies EJR-1 under the interaction settings. Example 4 give a detailed example for such a scenario.

Example 4 (EJR-1 in the interaction settings)**.** *Given PB instance with 8 projects split to partition: $z_1 = \{p_1, \ldots, p_4\}, z_2 = \{p_5, \ldots, p_8\}$. There is a single voter with a budget of 4.*

In addition, the projects in z_1 are total substitutes i.e. the first worth 1 and the rest 0, each cost 1. All projects in z_2 worth 1 and cost $1 + \epsilon$.

It is easy to see that the single voter is (α, z_1)-cohesive for $\alpha \equiv 1$, so $\alpha(z_1) = 4$, therefore EJR-1 require that either $u_i(W) \geq \alpha(z_1)$ or there is some project $p^* \in z_1$ such that $u_i(W \cup \{p^*\}) > \alpha(z_1)$.

There are three cases for W:

- *W contains at no projects from z_1. Then W contains at most 3 projects so even with an additional project p^* we have $u_i(W) < 4$ and $u_i(W \cup \{p^*\}) \leq 4$.*
- *W contains at least one project from z_1 but not all. Then W is still blocked by z_1, as $u_i(W \cup \{p^*\}) = u_i(W) \leq 3 < \alpha(z_1)$.*
- *$W = z_1$. Then $u_i(W) = 1 < \alpha(z_1) = 4$.*

The last case demonstrates a big issue where an outcome can block itself as α only consider singletons while the utility affected from projects interaction. Even if we disallow this, in the last case we have $T = \{p_5, p_6, p_7\}$. Our single voter is (α, T)-cohesive for $\alpha \equiv 1$, and $u_i(W) = 1 < \alpha(T) = 3$.

While this example show a case where EJR-1 outcome does not always exist, we can do a simple modification to show that EJR-1 and EJRI-1 are in-comparable. consider the same example, but the utility for all projects in z_1 are 1, and the marginal utilities for projects in z_2 grows exponentially (with base 10). This time, the only outcome to satisfy EJR-1 is z_1 (z_1 guarantee utility of 4 and z_2 guarantee utility of 3 according to EJR-1) while EJRI-1 is satisfied by choosing 3 projects from z_2, guaranteeing utility of 111. As can be seen different outcomes satisfy EJR-1 and EJRI-1 without any overlapping.

C Partition Interaction Equal Shares

In section 4.2 we saw that IES holds EJRI-1 for substitute projects, however this does not hold anymore for general interactions. For this reason, we suggest a variation of IES called *Partition Interaction Equal Shares* (PIES). This mechanism is same as IES, but with a preprocessing step: for each part $z \in Z$ and every $T \subseteq Z$, add a new project p_T and remove all original projects. We set $cost(p_T) := cost(T)$ and $u_i(p_T) := u_i(T)$ for all $i \in V$. Once we have the new set of projects the aggregation will be performed similarly to IES, with the difference that at the end of each iteration where a project p_T is chosen, all projects $p_{T'}$ with $T \cap T' \neq \emptyset$ are removed.

When running PIES we have a larger amount of projects which is exponential in $|z^*| := \max_{z \in Z} |z|$. This means that at each iteration we need to iterate over $O(|Z|2^{|z^*|})$ projects instead of only $O(M)$, making PIES less efficient compared to ES and IES. However, it is likely to assume that each interaction set size is bounded by some relatively small value, which in this case we get that the aggregation will still run in reasonable time.

Proposition 6. *PIES holds EJRI-z for any interaction function.*

We give intuition for the proof. When using PIES, we look at a PB instance where every combination of projects in each interaction set is represented as a project. In this scenario, all interaction functions actually behave as functions for substitute projects, this due to the fact that choosing a project will remove all other overlapping projects and the utility for any non-overlapping project must be lower otherwise PIES would have chosen the project that represent both projects. PIES behave similarly to IES which we shown to hold EJRI-1 when all interaction functions are proportional, but since in PIES case each project can represent several projects from the same interaction set, the "up to 1 project" becomes "up to 1 interaction set".

D Proportionality with Multiple Partitions

In Section 4 we saw that IES and PIES can achieve proportionality in our settings. In this section, we consider the scenario where each voter can submit a different partition, demonstrating that those methods fail to satisfy proportionality.

Example 5. *Given a participatory budgeting scenario with 3 voters $\{v_1, v_2, v_3\}$ and 16 projects $\{a_{1-3}, b_{1-3}, c_{1-3}, d_{1-3}, e_{1-4}\}$, where cost$(a_{1-3})$ = cost(b_{1-3}) = cost(c_{1-3}) = cost(d_{1-3}) = 1, cost$(e_{1-4}) = \frac{3}{2}$ and for each $i \in [1, 3]$ the voters approve the following:*

- $v_1 : \{(b_i), (c_i), (a_i, d_i)\}$
- $v_2 : \{(b_i), (a_i, c_i), (d_i)\}$
- $v_3 : \{(a_i, b_i), (c_i), (d_i)\}$

Where projects at the same parenthesis the voters want exactly one of them (the second project will have utility of zero). In addition, all voters approve e_{1-4} (without interaction).

The utility for all projects (expect for the interactions) is one and the total budget is $L = 9$. We will use IES (PIES works similarly) for aggregation with tie-breaking for the worst case (tie-breaking done for easier readability, it is possible to change the utility or cost by a small ϵ value and the outcome will not change without the need for tie-breaking).

At the first step projects $a_{1-3}, b_{1-3}, c_{1-3}, d_{1-3}$ are $\frac{1}{3}$-affordable, while projects e_{1-4} are $\frac{1}{2}$-affordable. Using tie breaking, IES will choose to fund project a_1, leaving each voter with budget of $\frac{8}{3}$.

After choosing project a_1, the utility of the voters update accordingly $u_1(d_1) = u_2(c_1) = u_3(b_1) = 0$, resulting that projects a,b,c becoming $\frac{1}{2}$-affordable, while the other projects remain the same. In addition, each voter have a budget of two remaining.

In similar manner, projects a_2 and a_3 will be chosen. This results with all projects being $\frac{1}{2}$-affordable as only two voters give utility > 0 for projects $b_{1-3}, c_{1-3}, d_{1-3}$, while the utility of e_{1-4} projects is still unchanged.

As there are not any interactions left, the utility for all projects will stay the same until the aggregation stops and projects will be chosen by tie-breaking. For this reason, we tie-break in favor of e_{1-4} and choosing to fund all of them, which result with exhausting the budgeting and getting an outcome of $W = \{a_{1-3}, e_{1-4}\}$ which give utility of 7.

Lets note the projects set $T = \{a_{1-3}, b_{1-3}, c_{1-3}\}$ and set of voters $S = \{v_1, v_2, v_3\}$ which are T-cohesive with $\alpha(c, B_t) = 1$, therefore there is at least one voter which should get utility of 9. However, the outcome utility is 7 and even when adding one more project will be 8, still lower than required. This is a violation of EJR-1.

An intuition to why IES does not hold EJR-1 in the example, is looking at the projects $a_i - d_1$ as a single interaction set where the interaction function can give a different value for different sets of projects and does not look only at amount i.e. it isn't indifferent to the which project is funded anymore. Next, note that the guarantee of EJR-1 for $(\alpha, T) - cohesive$ groups depends only by the utility of projects in T, however if the interaction functions is not indifferent to which project is chosen, there might be some project outside of T which hurts the utility of T.

E Welfare Worse-Case Analysis

In section 5.1 we saw the welfare ratio of ES in the additive settings, here we will dive into the reason we get such ratio and demonstrate it with an example.

When running ES (or one of its variations), in each iteration we search for the project that currently has the lowest qValue, however while it is practically calculated as the ratio between cost and utility, it does not necessarily mean that projects with high utility will be chosen. The reason for this is that the qValue should also be “fair", which is reflected by higher qValue when supporters' budget is constrained.

This behavior mean that ES variations will prioritize projects where there is larger agreement about how much utility this project is worth i.e. projects with lower variance over their utility. Therefore, ES can result with outcomes with worse social welfare to maintain the fairness. This downside can become more noticeable when using PIES. The reason for this that when considering a set of projects it might have a high utility for many voters (including complementary projects), but even one voter which does not agree with it and give it a low utility it will have a very high qValue. Therefore, will miss the option to choose projects with very high utility.

To demonstrate this behavior, lets look at a scenario with 5 voters, 2 projects that cost 10 and budget of 10. All of the voters approve the first project with utility of 2 and the second project with utility 10 except for one voter which give it utility of 2 (note we use additive utilities). The first project will be 1-affordable, while the second project with much higher welfare will be 2-affordable. In both cases all 5 voters will use their entire funds, however the project with higher total utility will have higher qValue, thus the other project will be funded. This behavior will be further demonstrated in experiment at Section 5.2.

This scenario can be further extended such that the utility of the second project is increased from 10. No matter how much we increase this value, as long there is one voter with low utility, the first project will still be chosen resulting with the same welfare while the optimal social welfare keep increasing.

F Simulations Full Results

This section present the results from the experiments described in Section 5.2 for all types of interaction functions in Figures 4-12. We remind the reader the six types of interaction functions:

1. Type 1 (minimal substitutes) - the voter prefers one non-substitute project over the entire set of substitute projects i.e. the utility for the first project is 1, while any additional project gives the utility of $1/m$
2. Type 2 (harmonic) - the marginal utility decrease by harmonic series, formally: $f(i) = \sum_{j=1}^{i} \frac{1}{j}$
3. Type 3 (exponential sub) - the marginal utility decreases exponentially, formally: $f(i) = \sum_{j=1}^{i} \frac{1}{1.5^{j-1}}$
4. Type 4 (extreme complementary) - the voter significantly prefer at least two projects and want as many as possible, formally the marginal utility for the first project is 1 and 50 for the rest.
5. Type 5 (linear) - the marginal utility increase linearly, formally: $f(i) = \sum_{j=1}^{i} j$
6. Type 6 (exponential comp) - the marginal utility increases exponentially, formally: $f(i) = \sum_{j=1}^{i} 1.5^{j-1}$

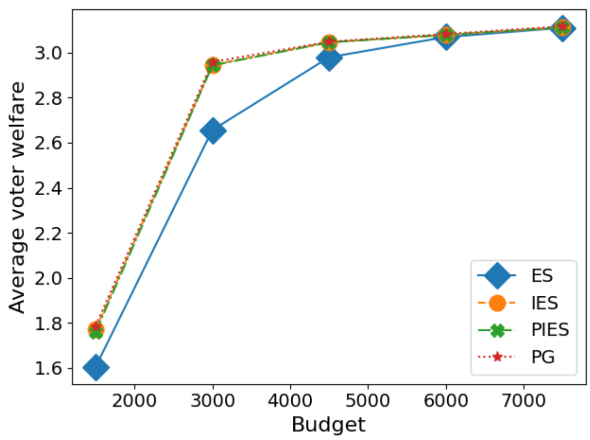

Figure 4: Average voter welfare over 1000 simulations, given only minimal substitutes voters.

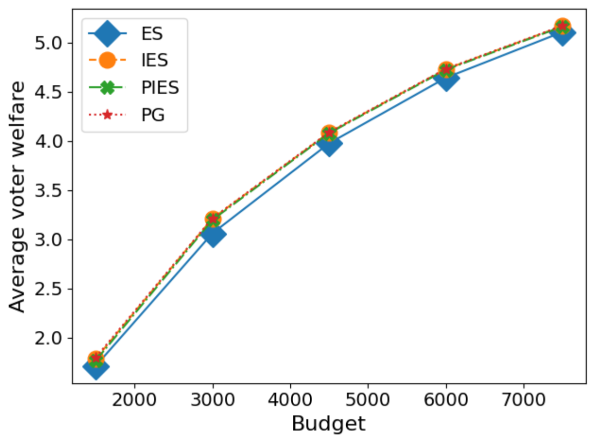

Figure 5: Average voter welfare over 1000 simulations, given only harmonic voters.

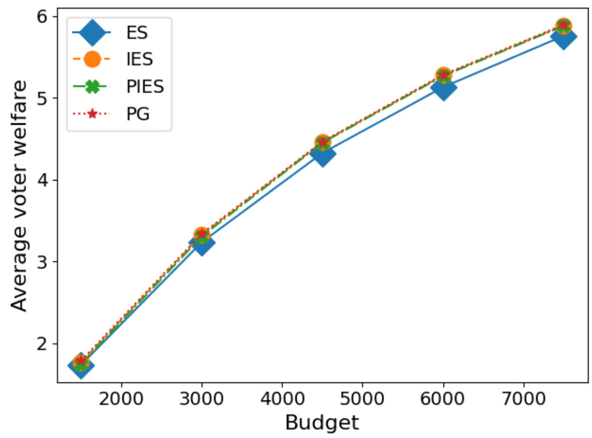

Figure 6: Average voter welfare over 1000 simulations, given only exponential sub voters.

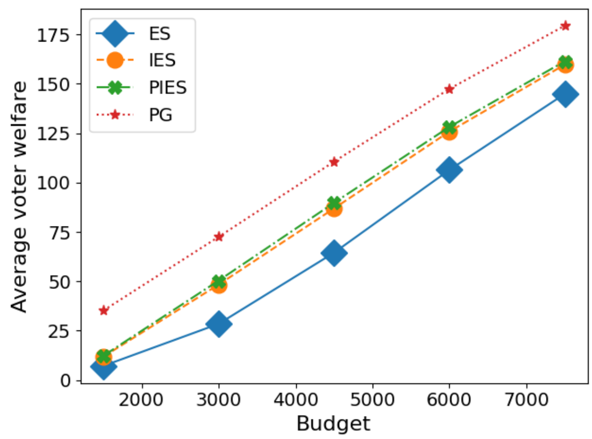

Figure 7: Average voter welfare over 1000 simulations, given only extreme complementary voters.

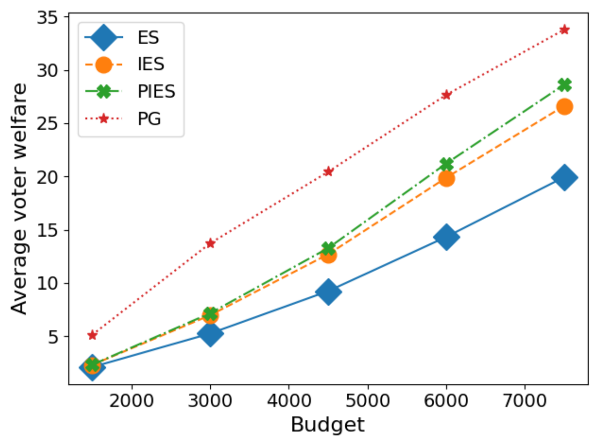

Figure 8: Average voter welfare over 1000 simulations, given only linear voters.

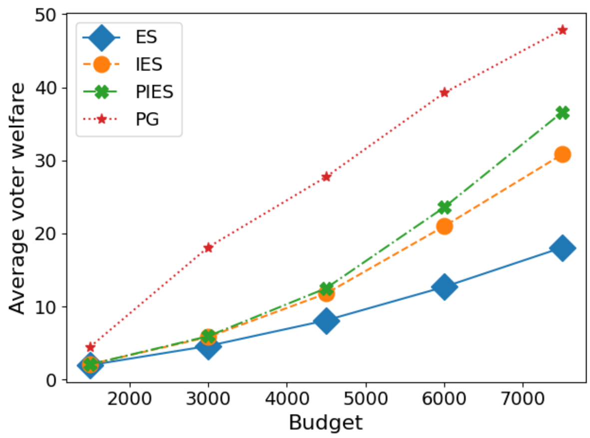

Figure 9: Average voter welfare over 1000 simulations, given only exponential comp voters.

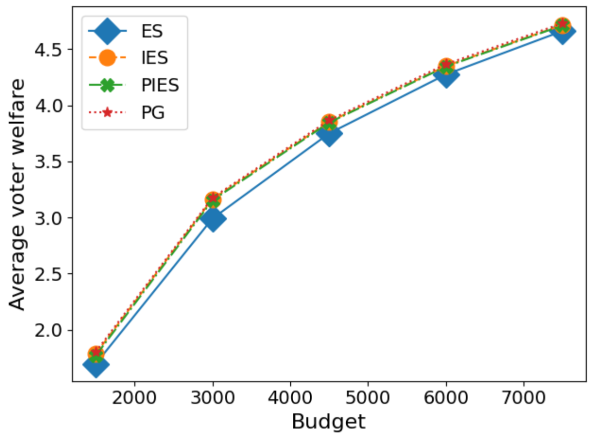

Figure 10: Average voter welfare over 1000 simulations, voters have mix of substitution interactions i.e. minimal substitutes, harmonic and exponential sub.

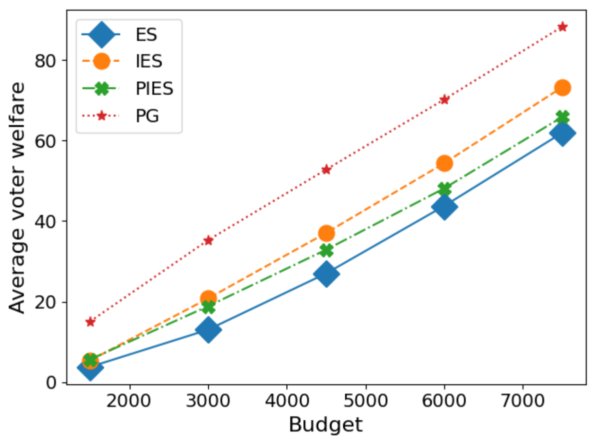

Figure 11: Average voter welfare over 1000 simulations, voters have mix of complementary interactions i.e. extreme complementary, linear and exponential comp .

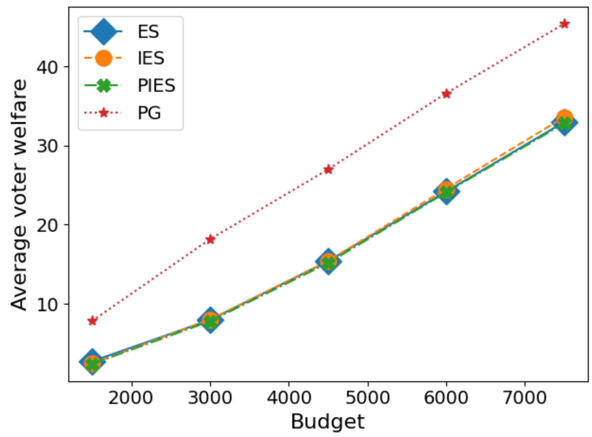

Figure 12: Average voter welfare over 1000 simulations, voters have mix of all possible interactions.