A Non-Jury Theorem when Voters Can Abstain

Ganesh Ghalme[a,*] **and Reshef Meir**[b]

[a]Indian Institute of Technology Hyderabad, India
[b]Technion-Israel Institute of Techology Haifa, Israel

Abstract. The well-known Condorcet's Jury theorem posits that the majority rule selects the best alternative among two available options with probability one, as the population size increases to infinity. We study this result under an asymmetric two-candidate setup, where supporters of both candidates may have different participation costs.

When the decision to abstain is fully rational i.e., when the vote pivotality is the probability of a tie, the only equilibrium outcome is a trivial equilibrium where all voters except those with zero voting cost, abstain. We propose and analyze a more practical, boundedly rational model where voters overestimate their pivotality, and show that under this model, non-trivial equilibria emerge where the winning probability of both candidates is bounded away from one.

That is, victory is not assured to any candidate in any non-trivial equilibrium, regardless of population size and in contrast to Condorcet's assertion.

1 Introduction

Consider a population of N voters voting over two alternatives A and B, with A being the *better alternative* according to some predefined criterion. Consider further that the preference of each individual voter is determined independently by an outcome of a coin toss with bias $p > 0.5$ in favour of the better alternative. That is, each individual voter supports alternative A with probability p and B with the probability $1 - p$. Under this setting, the famous Condorcet's Jury Theorem states that the majority rule selects candidate A with probability tending to one when population size increases to infinity. [1]

An implicit assumption in Condorcet's theorem is that *everyone votes*, or at least that the decision to vote does not depend on one's preference over alternatives. In contrast, in many practical situations such as political elections, or a local or national referendum, abstention is found to be a common and prominent phenomenon. For instance, the voter turnout in United States presidential elections has been around 52%-62% over the past 90 years [19]. Abstention is also observed to be a significant phenomenon in small-scale laboratory experiments [2, 17].

From a rational, economic point of view, the surprise is not that some voters abstain, but that they vote at all a.k.a. 'the paradox of voting'. As Anthony Downs claimed already in 1957, a rational voter weighs the benefit of voting (which realizes only if the voter is pivotal) against the cost. When the size of the electorate is large, the expected benefit derived from affecting the outcome of the election (i.e. being pivotal) is too small to induce voting from a significant fraction of voters [3] giving rise to the paradox of voting.

While Condorcet's result holds under the assumption that everyone votes, Down's theory of rational voting suggests that only near-zero cost voters vote in large-scale elections. However, the data from large-scale elections (US presidential elections, for instance) suggests that a significant fraction of voters vote, contradicting the prediction by the rational voting model in practice. This paper evaluates Condorcet's result in a two-candidate election setting where rational voters make a strategic choice to abstain from voting.

Our goal in this paper is to understand the equilibria that arise from a plausible heuristic-based abstention model. The most important questions that we focus on are (1) How many equilibrium points are induced by the abstention model and where are they located? (2) Does the winning probability of the better/popular alternative/candidate approach 1 as predicted by Condorcet's Jury theorem in any of these equilibria? (3) What happens to the equilibrium points as population size increases? (4) Does multi-round voting ensure that a better alternative is chosen with higher probability over single-round voting?

We now present some important models for voter turnout presented in the literature and then move on to our proposed voter turnout model that depends on heuristic-based perceived pivotality of individual votes.

The Calculus of Voting model: Originally proposed by Downs [3] and later developed by Riker and Ordeshook [18], this model attributes each voter's decision to abstain from voting to expected cost-benefit analysis. Let, for a voter i, p_i denotes the perceived *pivotality* of her vote, $\mathtt{V}_i$ denotes the personal benefit she receives if her preferred candidate wins an election, $\mathtt{D}_i$ denotes the social benefit she receives by performing a civic duty of voting and $\mathtt{G}_i$ denotes costs of voting she incurs. These costs include the cost of obtaining and processing information and the actual cost of registering and going to polls (see also [1] for discussion of voting and rational choice). A voter i votes if and only if

$$p_i \cdot \mathtt{V}_i + \mathtt{D}_i \geq \mathtt{G}_i. \tag{1}$$

The calculus of voting model considers p_i to be the probability that a single vote would change the outcome a.k.a. probability of an event that the outcome of an election is a tie.[2] The tie probabilities are derived from the aggregated stochastic votes, and thus the pivot computation and subsequent equilibrium analysis quickly become intractable as the number of individual voters increases.

Myerson and Weber studied voting equilibria in a 3 or more candidate elections by fixing a distribution over preferences and consider-

* Corresponding Author. Email: ganeshghalme@ai.iith.ac.in

[1]Latest version of the paper is available online at https://arxiv.org/pdf/2408.00317.

[2]We will consider that the number of voters is odd.

ing a large population sampled from this distribution, so that candidates' scores are multinomial variables [15]. While tie probabilities vanish in the limit, they observe that the *relative* tie probabilities for each pair of candidates can still be compared and thus voters' best responses are well defined. In a more recent work, Myerson [14] suggested another relaxation by approximating candidates' scores with Poisson distributions.

Crucially, in the above models, the tie probabilities used for voters' strategic calculations are derived directly from the vote distribution, either exactly or approximately. When the size of the electorate is large, the tie probability $p \approx 0$. If voting is costly, these models predict very low turnout—essentially that only the zero-cost voters vote when the population increases to infinity.

Heuristic pivot probabilities: Some models assume that the voters act based on estimated or even completely wrong beliefs. One such heuristic is derived by Osborne and Rubinstein's *sampling equilibrium* [16], where each voter estimates candidates' scores based on a small random sample of other voters. That is, even with a large population when voters compute their beliefs based on limited information, there is a non-negligible fraction of voters who believe they are pivotal.

Non-probabilistic uncertainty: Other models stir away from probability calculations and consider other voters' heuristics, based on dominated actions, minmax outcomes or regret [1, 7, 10, 12]. The *margin* of victory plays a major role in the voter's perceived pivotality. Controlled experiments on voters' response to poll information show that *strategic* voting is more frequent when the winning margin is small [11, 7]. It is also observed that the decrease in participating voters is very slow and not consistent with standard models of rational voting behaviour [2, 5]. Fairstein et al. [5] show that voters' actions are more consistent with various heuristics based on the margin than with 'rational' utility maximization models. The key point in these models is that voters form a heuristic belief of their pivotality based on the voting profile, and that an equilibrium means the belief justifies itself [1, 9].

Analyzing every such model separately would be tedious and leave us with an isolated set of narrow results. Instead, we identify the main factors common to all of these models, and suggest a flexible framework that captures a wide range of possible rational and boundedly-rational behaviours, without committing to one model in particular. We still consider that the voters seek to maximize their own expected utility as in the calculus of voting model (Eq. (1)). However, in contrast to the calculus of voting model, in our model voters decide to abstain based on *heuristically* estimated perceived pivotality. This estimate is also a function of the margin of victory as in the latter models we mentioned. Our proposed heuristic model for computation of perceived pivotality is inline both with empirical [4, 6] and experimental [2, 5] findings, and with the models developed therein that aim to explain high turnout in elections.

We emphasize here that we do not claim to provide a new universal model of strategic voting. Rather, we want to capture a broad class of models that share similar properties (dependency on margin and population size) in order to provide results that are not model-specific.

Contributions We show that the set of equilibrium points induced by proposed abstention model consists of equilibrium points predicted by the fully rational (such as calculus of voting) model that satisfy Condorcet's Jury Theorem (CJT) and some more *non-trivial* equilibrium points. Our results show that there are *non-trivial* equilibria where the popular candidate is more likely to win, but the winning probability is bounded away from 1. This probability, perhaps surprisingly, depends only on the distribution of voting costs in the population. The bottom line of this paper is the result that under a plausible abstention model, induced non-trivial equilibrium points evade both Down's paradox of voting (only a small fraction of homogeneous voters vote) as well as CJT (the win probability of popular candidate approaches 1).

2 Model

We study a two-candidate (referred to as A and B) election with N voters. Each voter is either a supporter of A (prefers candidate A, i.e. $A \succ_i B$) or a supporter of B. We adopt the classical calculus of voting model as follows. We denote as $c_i := \max\{0, \frac{\mathtt{G}_i - \mathtt{D}_i}{\mathtt{V}_i}\}$ the *effective* cost of voting for voter i with $\mathtt{G}_i$, $\mathtt{D}_i$ and $\mathtt{V}_i$ as defined in Eq. (1).

The core supporters (voters with zero effective voting cost) derive more utility from voting for their preferred candidate than costs incurred in participating in the voting process. Voters with non-zero effective costs, on the other hand, vote only if the perceived pivotality of their vote exceeds the voting cost. For mathematical convenience, we normalize the effective cost of voting to lie between 0 and 1.

A better alternative in a two candidate election setting is the most popular candidate under Condorcet's Jury theorem, i.e. the candidate who wins an election with probability one when the size of the voting population tends to infinity. Note that except in a case where both candidates have equal support (aka when $p = 0.5$) such a candidate always exists.

We consider that the effective cost of voting and the preference of voter i as an independent sample from a commonly known joint distribution $\mathcal{D}$ over $[0, 1] \times \{A, B\}$ and is denoted by tuple (c_i, T_i). Without loss of generality, we consider that more voters support A in expectation a.k.a. A is the better/popular candidate. In an objective sense, A is the winning candidate (in the limit) if the preferences of everyone—including the abstaining voters—in the population were aggregated. That is, A would have won the election with the probability tending to one had every voter voted irrespective of her voting cost.

Prior to voting, every voter, given a cost distribution $\mathcal{D}$ and electorate size N, determines the perceived pivotality of her vote. Each voter i then privately realizes the tuple (c_i, T_i) sampled independently from $\mathcal{D}$. With the knowledge of $N, (c_i, T_i)$ and $\mathcal{D}$, a voter i then makes a choice of whether to vote or to abstain. This choice crucially depends on the perceived pivotality p_i of her vote. We consider that p_i is a function of the expected[3] number of votes for each candidate and the expected winning margin. Note that since $\mathcal{D}$ and N are fixed and commonly known we have, $p_i = p$ for all $i \in [N]$. In particular, the perceived pivotality p_i does not depend on the voters' type T_i. However, note that her choice to abstain may depend on her type.

Abstention Model: A voter i votes if $p \geq c_i$ and abstains otherwise. As mentioned previously, the *perceived pivotality* p denotes the voter's belief about the extent to which a single vote (her vote) is important. Thus a voter votes if she *perceives* herself as sufficiently pivotal. While c_i is privately known, p is a common knowledge and this value depends only on the aggregate voting profile.

Support Functions: Given a cost threshold c, let the random variable $X_A^{(i)}(c) := \frac{1}{N-1} \sum_{j \neq i} \mathbb{1}[c_j \leq c \text{ and } A \succ_j B]$ denote fraction of supporters—other than voter i—of candidate A having the voting

[3]Expectation is taken with respect to distributions $\mathcal{D}_A$ and $\mathcal{D}_B$.

cost at-most c and similarly, define $X_B^{(i)}(c) := \frac{1}{N-1}\sum_{j\neq i}^{N-1}\mathbb{1}[c_j \leq c \text{ and } B \succ_j A]$. Note that, as $X_T^{(i)}(c)$ does not depend on her type T_i and her private cost c_i and since every agent is exposed to the same knowledge, we have, $X_T(c) := X_T^{(1)}(c) = X_T^{(2)}(c) = \cdots = X_T^{(N)}(c)$ for all $T \in \{A, B\}$. Let $s_T(c) = \mathbb{E}[X_T(c)]$ be the expected fraction of supporters of candidate T with cost at-most c. We call $s_T(.)$ the *support function* of candidate T.

We consider that $s_A, s_B : [0,1] \to [0,1]$ are continuous, non-decreasing functions such that $s_A(c) + s_B(c) \leq 1$ for all $c \in [0,1]$ with $s_A(1) + s_B(1) = 1$ (see Figures. 1 and 4). An election instance $\mathcal{I}$ is thus given by tuple $\langle N, s_A, s_B\rangle$.

Given $c \in [0,1]$, the expected number of voters with effective voting cost at most c is given by

$$n(c) = (s_A(c) + s_B(c))N. \tag{2}$$

We remark here that $n(c)$ is a *point estimate* of the voter turnout with cost threshold c. Similarly, the estimated margin of victory m is derived directly from the number of active voters n and the support functions as

$$m(c) = \frac{|s_A(c) - s_B(c)|}{s_A(c) + s_B(c)}. \tag{3}$$

We omit the argument c from m and n whenever it is clear from context. A Perceived Pivotality Model (PPM) is a decreasing function $p : \mathbb{N} \times [0,1] \to \mathbb{R}$, mapping the number of active voters n and the estimated margin of victory m to a perceived pivot probability.

As mentioned earlier, perceived pivotality is each agents' subjective assessment of the likelihood that her vote is pivotal. In next section, we briefly overview fully-rational and heuristic-based PPMs studies in literature and propose tie-sensitive PPM studied in the rest of the paper.

3 Perceived Pivotality Models

In models that aim to capture fully rational behaviour, p represents the probability that a given voter's vote is pivotal; i.e. the probability that an individual vote will be used to determine the winner of the election. We first describe models where the perceived pivotality exactly or approximately represents the tie probability, and thus approaches 0 as the population increases to infinity.

Fully Rational model

Definition 1 (Strongly Vanishing PPMs)**.** *We say that a PPM has strong vanishing pivotality if* $\lim_{n\to\infty} p(n, m) = 0$ *for all* $m \geq 0$.

Perhaps the most common pivot probability model is a Binomial model, corresponding to a known number of voters n who choose independently whether to vote for A or B. This is exactly the model used in Condorcet's Jury Theorem and in early Calculus of Voting models [18].

Example 1. *(Binomial PPM) Let $\mathcal{I}$ be a given election instance with* $s_A(0), s_B(0) > 0$. [4]

$$p(n, m) = \Pr_{x \sim Bin\left(n, \frac{s_A(c)}{s_A(c)+s_B(c)}\right)} (x = \lfloor n/2 \rfloor). \tag{4}$$

For large n using Stirling's approximation, we have

$$p(n, m) \cong \sqrt{\frac{2}{\pi n}}((1+m)(1-m))^{(n/2)} < \frac{1}{\sqrt{n}}$$

We note that the pivot probability decreases at rate $1/\sqrt{n}$ for $m \approx 0$, and much faster for a fixed positive margin. Thus the model suggests that in a large election only extremely low-cost voters would vote. That is, in large elections the outcome is determined only by the fraction of core supporters of the candidates.

The Binomial model introduces a dependency between candidates' scores that is difficult to work with. Hence, a later model by Myerson [13] suggested drawing the scores of each candidate independently from a Poisson distribution.

Example 2. *(Poisson PPM): Let $\mathcal{I}$ be a given election instance. The Poisson PPM model considers the perceived pivotality as the probability that an equal number of supporters are drawn from Poisson distributions with parameters $N \cdot s_A(c)$ and $N \cdot s_B(c)$. That is,*

$$p(n, m) = \Pr_{\substack{k_A \sim Poisson(s_A(c)N) \\ k_B \sim Poisson(s_B(c)N)}} (k_A = k_B) \tag{5}$$

Conceptually, the Poisson model is more appropriate in situations where voters can abstain (as the total number of active voters is not fixed). However $p(n, m)$ behaves very similarly to the Binomial model, and for our purpose they are almost the same (as all Strongly Vanishing PPMs are).

Tie-Sensitive model

A more cognitively plausible assumption is that voters *consider themselves pivotal* if the margin is small enough, regardless of the number of voters. The dependence of such tie-sensitive models on the margin m and number of active voters n is captured by the following two properties. We can see these properties as an abstraction of various bounded rational behaviors described in the introduction.

Property 1. *Given $m > 0$, there exists $n_0 \in \mathbb{N}$ such that $p(n, m) \propto \frac{1}{\sqrt{n}}$ for $n \geq n_0$.*

Property 1 captures the dependence of perceived pivotality on the population size, which is aligned with the fully rational models.[5]

In contrast, the heuristic dependence of PPM as decreasing in the margin m, diverges from the rational models:

Property 2. *There exists $\alpha \in (0, \infty)$ such that for any fixed $n \in \mathbb{N}$, $p(n, m) \propto \frac{1}{m^\alpha}$.*

The simplest model that satisfies both properties is obtained by directly combining the dependency on n and m:

Definition 2 (Tie-sensitive PPM)**.** $p(n, m) = \min\{1, \frac{1}{m^\alpha \sqrt{n}}\}$ *for some* $\alpha > 0$.

For theoretical analysis, we assume in the remainder of this section that $\alpha = 1$ (linear margin PPM), and extend these results using simulations to other values of α in Section 8. We are interested in characterizing equilibria as $N \to \infty$.

[4] We assume that at least one of $s_A(0)$ and $s_B(0)$ is strictly positive.

[5] The $\sqrt{n}$ reflects the standard deviation of summing n independent variables.

Equilibiria

Definition 3. *Given an election instance $\mathcal{I}$ with PPM $p(.,.)$, we call $c \in [0,1]$ an equilibrium if*

$$c = p(n(c), m(c)). \tag{6}$$

We distinguish the equilibrium points into two categories: trivial and non-trivial. A **trivial equilibrium** point is characterized by the condition that as the population size increases to infinity, the equilibrium point converges to zero. We say that an election equilibrium admits a Jury theorem if the popular candidate emerges as a clear winner in the limit. That is, $\Pr(\text{A wins}) \to 1$ as $N \to \infty$.

Next we define pivot point as a cost value where both the candidates have equal support from the voters with costs atmost c.

Definition 4 (Pivot point)**.** *A cost value $c \in (0,1)$ is called a pivot point if $s_A(c) = s_B(c)$.*

Note that the set of pivot points may be empty. We now show that in this case, the trivial equilibrium is the unique equilibrium under tie-sensitive PPM. We formalize this argument in our first result.

Proposition 1. *Let $\mathcal{I}_N$ be an election instance with N voters and positive, continuous and increasing functions s_A, s_B defined over $[0,1]$ such that $s_A(c) > s_B(c)$ for all $c \in [0,1]$. Also let $c^0(N)$ be an equilibrium cost threshold (Eq. (6)) induced by a tie-sensitive PPM, then $\mathbb{P}(A\ wins) \to 1$ as $N \to \infty$.*

The proof of Proposition 1 follows directly the fact that since s_A and s_B do not intersect, $m(c) > 0$ for all c. Hence for every $c > 0$, there is a large enough N such that at least $(s_A(0) + s_B(0)) \times N$ voters vote. This fact along with Property 1 drives the perceived pivotality of the individual vote to 0 causing the equilibrium cost also to decrease to 0 in the limit.

In a more interesting case where s_A and s_B intersect, we argue that in addition to the trivial equilibrium around $c = 0$, two more nontrivial equilibria emerge—each on either side of pivot point $\hat{c}$ (see Figure 1 for illustration with linear support functions). Interestingly, these nontrivial equilibria admit 'non-Jury' result as the winning probabilities of any candidate (candidate A in Figure 1) is bounded away from 1. We now prove this formally.

4 A Non-Jury Theorem with Linear Support

Let $\mathcal{C}$ be the set of all pivot points. We begin by showing that for large values of N, non-trivial equilibria emerge around the pivot point and converge to the pivot point as N goes to infinity.

Proposition 2. *Let $\mathcal{I}_N$ be an election instance with N voters and linear, positive, continuous and increasing functions s_A, s_B defined over $[0,1]$ such that $s_A(\widehat{c}) = s_B(\widehat{c})$ for some $\widehat{c} \in (0,1)$. Under the tie-sensitive PPM with $\alpha = 1$, there are three equilibria $c^0(N) < c^-(N) < \hat{c} < c^+(N)$, such that*

1. $\lim_{N\to\infty} c^0(N) = 0$, *and*
2. $\lim_{N\to\infty} c^-(N) = \lim_{N\to\infty} c^+(N) = \widehat{c}$.

The proof of Proposition 2 is given in the Appendix. We remark that the equilibria $c^-(N)$ and $c^+(N)$ are not symmetric: only c^+ is stable (see arrows at top of Fig. 1 and Section 5); and winning probabilities behave differently (Section 8). We are now ready to prove a non-Jury result for the linear support setting.

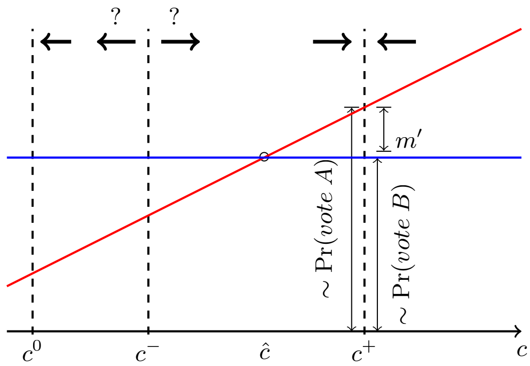

Figure 1: The pivot point $\hat{c}$ is marked by a circle at the intersection of the support functions, with the two non-trivial equilibria on its sides (dashed lines). For the upper equilibrium c^+, the probability of a random voter to vote A or B is proportional to $s_A(c^+)$ and $s_B(c^+)$, respectively. The m' is proportional to the margin of victory. The bold arrows above indicate that c^+, c^0 are stable equilibria whereas c^- is often not stable.

Theorem 3 (A non-Jury Theorem)**.** *Under the conditions of Proposition 2, in any nontrivial equilibrium there is a constant $\beta < 1$ such that the winning probability of either candidate is at most β.*

Proof. Let $\widehat{c} \in (0,1)$ be the intersection point, i.e. $s_A(\widehat{c}) = s_B(\widehat{c})$. We have from Theorem 2 that for sufficiently large N, there are exactly two non-trivial equilibria around the pivot point. We prove the stated result for each of the equilibrium points separately.

The right side equilibrium i.e. ($c^+ > \widehat{c}$): This case is depicted in Figure 1. Let μ be the expected number of votes received by candidate A i.e. $\mu = \Pr(vote\ A) \cdot n(c^+)$ where $\Pr(vote\ A)$ is the probability that a randomly selected voter votes for candidate A.

We prove the non-Jury result by showing that the absolute difference between the votes required to win an election $n(c^+)/2$ and the expected number of votes μ in favour of candidate A are close (see Figure 2). We first bound $\Pr(vote\ A)$.

Claim 4. *Let $\Pr(vote\ A)$ denote the probability that a randomly selected voter $i \in [N]$ votes for candidate A under an equilibrium c^+ i.e. $\Pr_{\mathcal{D}}(A \succ_i B \cap c_i \leq c^+)$. Then*

$$\frac{1}{2} + \varepsilon_1/\sqrt{N} \leq \Pr(vote\ A) \leq \frac{1}{2} + \varepsilon_2/\sqrt{N} \tag{7}$$

Where $\varepsilon_1 := \sqrt{s_A(\widehat{c})/2}$ and $\varepsilon_2 := 1/4\widehat{c}s_A(\widehat{c})$.

We have from Claim 4 that

$$\left|\frac{n(c^+)}{2} - \mu\right| = n(c^+)\left|\frac{1}{2} - \Pr(vote\ A)\right| \leq \varepsilon_2\sqrt{n(c^+)} \tag{8}$$

Furthermore, the winner is chosen by an election determined by a Binomial random variable with parameters $\Pr(vote\ A)$ and $n(c^+)$. The standard deviation of this random variable is given as

$$\begin{aligned}\sigma &= \sqrt{n(c^+)\Pr(vote\ A)(1 - \Pr(vote\ A))} \\ &\geq \sqrt{n(c^+)}\sqrt{(0.5 + \varepsilon_1/\sqrt{N})(0.5 - \varepsilon_2/\sqrt{N})}\end{aligned}$$

For $N \geq \max(16\varepsilon_1^2/3, 64\varepsilon_2^2/9)$ we have

$$\sigma \geq \sqrt{n(c^+)}/4. \tag{9}$$

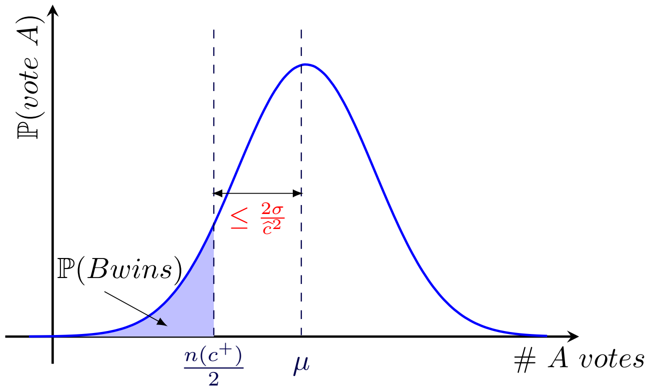

Figure 2: An illustration of non-Jury theorem under equilibrium c^+. The number of votes received by A follows a binomial distribution as shown by the blue curve. As N increases the σ gets smaller at rate $1/\sqrt{N}$, so the tail should get smaller as well; but on the other hand μ gets closer to $n^+/2$ at roughly the same rate, hence the non-Jury theorem.

Equations (8) and (9) together imply that $|n(c^+)/2 - \mu| \leq 4\sigma\varepsilon_2 = \frac{\sigma}{\widehat{c} s_A(\widehat{c})}$. We can further remove explicit dependence on the support function by observing that $\widehat{c} \leq 2s_A(\widehat{c})$ (proof in Appendix).

Observation 5. $\widehat{c} \leq 2s_A(\widehat{c})$.

That is, the difference between the mean and the threshold for A's victory is bounded by $\frac{2\sigma}{\widehat{c}^2}$ for a large value of N. Equivalently, the win probability of candidate B is at-least $1 - F(\frac{2\sigma}{\widehat{c}^2})$. This contradicts the Jury theorem (see Figure 2). Further, for a large value of N we can approximate the Binomial distribution by a Gaussian distribution with mean μ and standard deviation σ. The winning probability of candidate B is at-least $1 - \Phi(\frac{2\sigma}{\widehat{c}^2})$.

Interestingly the closeness between $n(c^+)/2$ and μ is determined by the position of the pivot point. For larger $\widehat{c}$ the two quantities are close and hence the probability that an unpopular candidate (candidate B) wins an election is comparatively large.

The left side equilibrium i.e. $(c^- < \widehat{c})$: The analysis of this case follows similarly to that for the right side equilibrium with the roles of A and B reversed.

Claim 6. *Let* $\Pr(vote\ B)$ *denote the probability that a randomly selected voter* $i \in [N]$ *votes for candidate* A *under an equilibrium* c^- *i.e.* $\Pr_{\mathcal{D}}(B \succ_i A \cap c_i \leq c^-)$*. Then*

$$\frac{1}{2} + \frac{\varepsilon_1}{\sqrt{N}} \leq \Pr(vote\ B) \leq \frac{1}{2} + \frac{\varepsilon_2}{\sqrt{N}}.$$

Where $\varepsilon_1 = \sqrt{\max(s_B(0), s_A(0))}/\widehat{c}$ *and* $\varepsilon_2 = \sqrt{2s_B(\widehat{c})}/\max(s_B(0), s_A(0))$.

Using similar calculations as in the right side equilibria, it is easy to see that the candidate B's probability of win is upper bounded by $\frac{\sigma\sqrt{2s_B(\widehat{c})}}{\widehat{c}\sqrt{\max(s_A(0), s_B(0))}}$.

The winning probability of candidate A is at-least $1 - \Phi(\frac{\sigma\sqrt{2s_A(\widehat{c})}}{\widehat{c}\sqrt{\max(s_A(0)+s_B(0)}})$ under c^-. $\square$

5 Stability of Equilibrium Points

In this section we show that the right side equilibrium i.e. c^+ is stable by showing that when the perceived pivotality is different across voter types, a constant fraction of voters have an incentive to participate (or abstain) such that the equilibrium c^+ is restored. Note that c^0 is clearly a stable equilibrium since the margin is a near-constant.

Proposition 7. *Let* $c_A \geq \widehat{c}$ *and* $c_B \geq \widehat{c}$ *be cost thresholds for voters of type* A *and* B *respectively, and perceived pivotality* p *is given as* $p = p(c_A, c_B) = p(m(c_A, c_B), n(c_A, c_B))$*. Then for any voting instance* I *with* N *voters, the best response of voters is such that* $c_A = c^+$ *and* $c_B = c^+$.

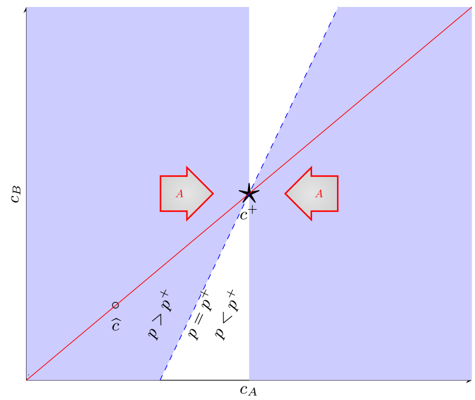

Figure 3: A schematic illustation of stability of the equilibrium point c^+. When $c' > c^+$ is an equilibrium estimate, the A supporters from right shaded region are incentivized to abstain whereas under $c" < c^+$ the A supporters from left shaded region are incetivized to participate.

We provide the intuition of the proof. The detailed proof is given in Appendix. Notice in equilibrium both types have the same cost threshold and hence the equilibrium points must be on the line $c_A = c_B$ as shown in Figure 3. Suppose first that $c_A > c_B$. In this case, the supporters of A shown in the shaded area (in Figure 3) to the right of c^+ are less pivotal and hence are incentivized to abstain. Similarly, when $c_A < c_B$, the A voters in the shaded region on the left side of c^+ are more pivotal and have an incentive to participate. The B supporters on the other hand will participate in the first case and abstain in the second case. Thus, the population of participating voters adjusts itself (i.e. $c_A \downarrow c^+$ and $s_B \uparrow c^+$ when $c_A > c_B$ and $c_A \uparrow c^+$ and $s_B \downarrow c^+$ when $c_A < c_B$) such that the equilibrium c^+ is restored.

The left-side equilibrium, on the other, hand may not always be stable. Consider $c" < c^-$ be an equilibrium with cost thresholds c_A and c_B with $c_A < c_B$. Then, the supporters of B are incentivized to participate as, with additional participation from supporters of B, the margin m increases, increasing the probability of win for B. The supporters of A, on the other hand are also incentivized to participate. An additional participation from A supporters would mean that m decreases. Hence stability of c^- depends on the relative increase in the participation from each type of agent.

The emergence of an unstable equilibrium between two stable ones (one of which is trivial) also occurs e.g. in markets with positive externalities [8]. In our case externalities behave non-monotonically (positive under $\hat{c}$ and negative above but the results are similar.

6 Beyond Linear Support

Our Non-Jury theorem (Theorem 3) immediately extends to any support functions as long as they are both linear in some environment of each intersection point $\hat{c}$ (such as piecewise linear non-decreasing function), around c_1, c_2). In case the support functions coincide over an interval, the same reasoning shows the equilibrium points will converge to the edges of the interval (see c_3^- and c^+ in Figure 4). One caveat though is that N needs to be sufficiently high so that the equilibria of each intersection point are within its linear environment.

Corollary 8. *The Non-Jury theorem holds under the linear margin PPM for piecewise linear support functions for sufficiently large N.*

We further conjecture that any well behaved cost function satisfies the Non-Jury theorem.

Conjecture 1. *The Non-Jury theorem holds for any sufficiently tie-sensitive PPM satisfying properties 1 and 2 and for any continuous support functions with bounded second derivative.*

By 'sufficiently tie sensitive' we mean that the dependency on m should be strong. E.g. for the tie sensitive PPM, our simulations show that nontrivial equilibria exist only when $\alpha \geq 0.5$.

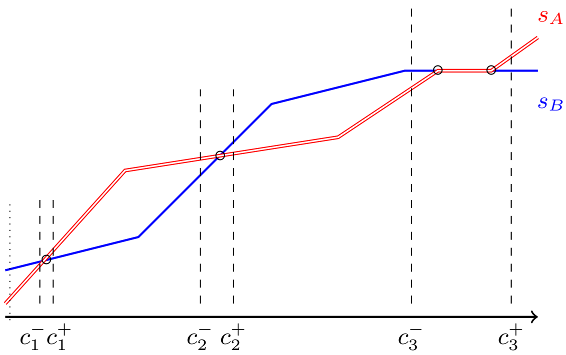

Figure 4: A schematic illustration of equilibria points with piecewise linear support functions. The dotted line marks the trivial equilibrium close to $c = 0$. We can see there are two nontrivial equilibria on the sides of each intersection point/interval of the support functions.

7 Inducing Unbiased Participation

In the previous section we saw that allowing everyone to participate may introduce multiple equilibria, including some where the less-popular candidate wins with higher probability, and where the probability of the popular candidate to win is bounded regardless.

We next explore a different idea: instead of allowing everyone to vote, we sample a sufficiently small group of voters (odd n), such that they are all guaranteed to perceive themselves as pivotal, and hence all of them vote. The result is unbiased voting, but with fewer voters. A small sample size also guarantees unique, stable equilibrium that favours the popular candidate.

Denote by $\varepsilon := s_A(1) - s_B(1)$, the expected margin under full participation, so $s_A(1) = (1 + \varepsilon)/2$ then $n \leq 1/\varepsilon^2$. We then need to select sufficiently small n so that $p(n, \varepsilon) \geq 1$. In the fully rational models, this can only occur for a single active voter. However under our tie-sensitive model we have

$$1 \leq p(n, \varepsilon) = 1/\varepsilon\sqrt{n} \tag{10}$$

so any $n \leq 1/\varepsilon^2$ will guarantee participation of the entire sample. The perceived margin provides sufficient information to bound the winning probability of popular candidate; independent of the shapes of support functions.

We now first consider a single round. This already provides us with a uniform lower bound of 0.84134 on the win probability of A when the margin ε goes to 0. The number 0.84134 is not coincidental but is the probability that a standard Normal random variable does not exceed a single standard deviation (See figure 5). Of course, the number is sensitive to the parameters of the exact PPM we use (as are the results in the previous sections), but not to the shapes of the support functions.

For larger margin, the win probability of A may be either higher or lower, and this depends mainly on the parity of $1/\varepsilon^2$ (where for odd numbers it is always higher).

Still, we would like to further improve the winning probability. One direction (which we do not explore in this work) is to sample a higher number of voters in an attempt to balance bias and quantity. Instead, we will explore the possibility of allowing multiple rounds.

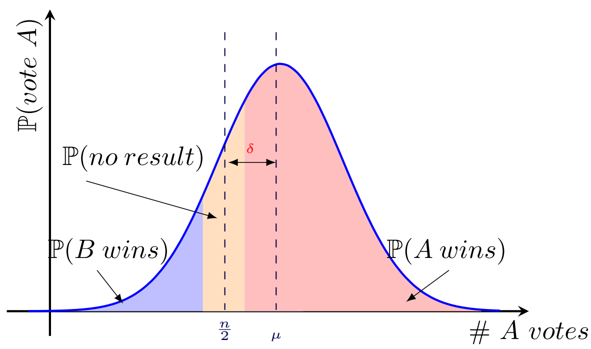

Figure 5: The figure illustrates win probability of the popular candidate in any given intermediate round. For majority, $\Pr(A\ wins) = 0.84134$ and similarly for supermajority with margin $m \times 0.3$ (shown by orange color) it is 0.7580 (the red area).

Multi-round voting Note that if we require a supermajority instead of a majority, the winning probability of A decreases, but not as much as the winning probability of B (see Fig 5). So in the extreme case, we could just try again and again until we get a unanimous vote. However such an extreme approach may substantially alter voters' behavior, which do not even know the number of rounds. We can thus limit voting to at most two rounds. Given an expected margin ε:

- Set a supermajority threshold $\varepsilon' < \varepsilon$;
- Sample $n = n(\varepsilon)$ voters;
- If the first round ends with a margin at least ε', the winner is determined and there is no second round;
- Otherwise, we run a second round with $n' = n(\varepsilon')$ voters and simple majority.

Note that the chances of the unpopular candidate B winning in either round are small, as the first round require a supermajority, and the second round, if occurs, has $n' > n$ voters.

8 Numerical Study with an Example

Example 3. *Suppose $s_A(c) = 0.1 + c/2$ and $s_B(c) = 0.4$. That is, B has 40% overall support, all of them core supporters and A has 60% overall support and a fraction of voters are distributed over cost range* $[0, 1]$.

Partial turnout equilibrium

Figure 6 demonstrates the Non-Jury Theorem (Thm. 3), showing that the winning probability of A in the stable equilibrium c^+ is bounded by $\beta \cong 0.955$, even as the total population size N is increasing.

Interestingly, the winning probability of B behaves differently and is *decreasing* towards the c^- equilibrium point. Hence B would always prefer a smaller fraction of the population to vote whereas the popular candidate prefers a large population to vote.

Varying the PPM parameters Figure 7 shows the win probability of candidate A under c^+ and for a fixed population size of 10^6 for different values of α. The larger value of α pushes the equilibrium point c^+ towards the right, increasing the win probability of the popular candidate. However, as shown in the blowup box, this probability is still bounded away from 1.

Unbiased Voting

In order to guarantee full turnout in our example, with its 0.2 margin, we could sample 25 voters for a winning probability of 0.846.

Two-round voting Suppose we impose a supermajority requirement of 14 out of 25 voters (i.e. a margin of 0.12). This means A is selected in the first round w.p. of $\sim 73\%$, and with a probability of 19% will proceed to a second round. However in that case the realized margin that voters observe is $\frac{14-11}{25} = 0.12$, which means in the second round we can sample 69 voters that are guaranteed to vote! Thus A's winning probability is 95% in the second round and 91% overall. This is still lower than the winning probability in limit equilibrium (~ 0.955), but with much fewer voters. Note that thousands of voters are required to reach a 91% winning probability in equilibrium without sampling.

In Fig. 8 we can see that as we consider instances with smaller margin, the winning probability of A remains stable under either 1 or 2 rounds (of course,the sample size is increasing). Finally, varying the α parameter in our PPM shows a curious phenomenon: for $\alpha < 1$ (meaning that the voters attribute lower weight to the margin in their estimation), the turnout does not increase high enough and the winning probability falls for low margins. The exact opposite occurs for $\alpha > 1$, thus experimental studies on actual voting behavior could be crucial for a better understanding of the best mechanism.

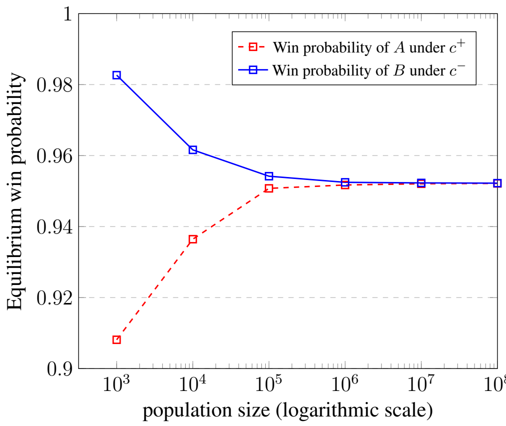

Figure 6: Win probability for different values of N under respective induced equilibria. The equilibrium win probability for a popular candidate A increases with N whereas it decreases for the unpopular candidate B.

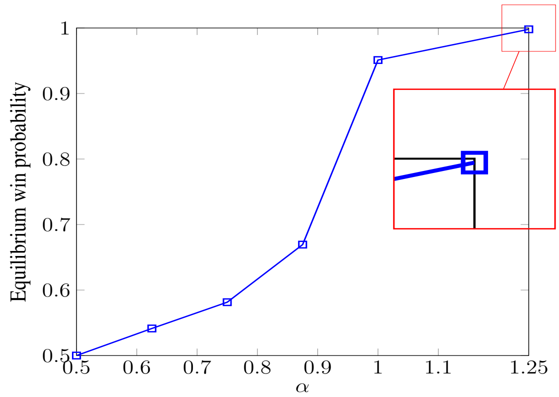

Figure 7: Win probability of candidate A under different values of α in $p(n,m) = \min(1, \frac{1}{m^\alpha \sqrt{N}})$. The blowup shows that win probability is still below 1 for $\alpha = 1.25$.

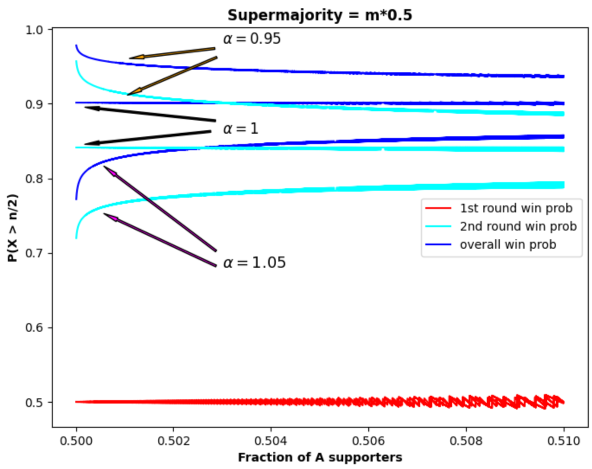

Figure 8: Win probability of A and its dependence on α in a two-round voting with supermajority rule (of margin*0.5) in the first round and the majority rule in the second round.

9 Conclusion

Our results show that under the boundedly rational PPM the equilibrium outcome does not guarantee a decisive win for any of the candidates even with an arbitrarily large electorate size. Interestingly, the proposed model also captures the equilibria induced in fully rational models (Binomial and Poisson models) as trivial equilibria.

Our boundedly rational model satisfies the property that both candidates enjoy almost equal support due to disproportionate abstention from their supporters. The *almost* equal support for candidates is a result of a trade-off between the following two factors; a) the candidate having majority support among high-cost voters faces more abstention. This increases the winning probability of the unpopular candidate against the popular candidate and b) voters are incentivized to vote depending on how many *other* voters vote. Hence, when a large fraction (overall) of voters abstain, the size of the voting population shrinks increasing pivotality and causing more voters to vote.

The number of votes on the ballot increases with the voting population, however, this does not benefit both the candidates equally; disproportionate increase in the support of unpopular candidates leads to anti-Jury results. It is interesting to see how and if this happens in practice. We also leave the sensitivity analysis and robustness of these results as future work. Finally, coming up with a fully rational model that explains both high turnouts and surprises in large elections is an interesting future problem.

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