Defining Compatibility for Moral Preferences: a Condition based on Suzumura Consistency

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Abstract. When applied within the domain of computational ethics, the task of aggregating ordinal preferences raises specific issues. Indeed, in this case, all ordinal preferences do not have the same status, depending on whether they are induced by agents or by moral principles: for the latter, they should not be contradicted, as this would compromise the principle's ability to provide morally satisfying preferences. This paper proposes a definition of compatibility between ordinal preferences in this context: it imposes that they can be combined into a single fused ordinal preference without performing any modification to any of them. To assess compatibility of a set of ordinal preferences, we prove a theorem that establishes a necessary and sufficient condition for this definition to hold: we show that it is equivalent to a modified version of the consistency property proposed by Suzumura (1976), which applies to a set of preorders. In addition, we propose an alternative and constructive proof for this property. This proof allows to define a procedure to determine whether a set of ordinal preferences is compatible: in the case it indeed holds, it allows to build a single total preference relation that contains all the considered preorders, i.e. compatible with all of them.

1 Introduction

Computational ethics, see e.g. [13] for a recent survey, globally aim at integrating ethical principles into decision-making systems. This can for instance involve designing systems to solve ethical problems, or ensuring that a decision-making system complies with ethical principles. For a given context that describes the state of the world and specifies a set of feasible decisions, these ethical principles can be viewed as processes which produce an order on the decisions, from the most to the least moral ones: they induce *moral preferences*, which are formalised as binary relations on a, usually finite, set of decisions.

Preference processing tools can therefore be used for computational ethics (see e.g. [10]). A common issue addressed by these tools is the aggregation of preferences, which is a problem found in particular in computational social choice. The aim is to reach a consensus between the various expressed preferences, even if it means contradicting some of them in the event of disagreement.

In the case of moral preferences, two types of preference must be distinguished: those that come from agents and those that come directly from the principles formalised in computational ethics. The latter have a special status: contradicting them undermines the principle's ability to provide morally satisfying preferences. This is why a disagreement between different ethical principles is more important than a disagreement between agents, and must be given special attention. Answering the question of their compatibility is therefore a first tool is order to determine the extent to which they can be used jointly. However, as discussed in Section 3, the definition of compatibility is not so intuitive as it goes beyond the usual properties expected of binary relations and preferences, as discussed in Section 3.

In this context, we propose a definition of compatibility for sets of preferences using extensions of binary relations: it requires that it is possible to combine them without any modification into a single preference relation. More precisely, a set of preferences is compatible if and only if there exists a total preorder, i.e. a transitive and total relation, which is an extension of all of them simultaneously. To assess compatibility of a given set of preferences, we prove theorems that establish a necessary and sufficient condition. We prove that the definition we propose is equivalent to a modified version of the consistency property proposed by [11], which applies to a set of preorders and provides a verifiable condition for the compatibility of a given set of preference relations. We also prove that this compatibility is equivalent to properties involving the union of the initial preferences. Thus, this compatibility corresponds to the case where the union is a satisfactory aggregation operator: a rare case in computational social choice as incompatibilities are assumed to exist among the considered preference relations.

Finally, in the case the considered ethical principles apply to finite sets of decisions, we propose an alternative proof without using Suzumura's theorems [11] by proposing the construction of a total preorder that is an extension of the preferences. The construction is decomposed into four steps. For the fourth one, we discuss the characteristics of various possible constructions, based on graph algorithms such as topological sort [9] or the longest path layering algorithm [3, 6]. We argue that the latter is more suited in an ethical context. Furthermore, we consider additional constraints that impose specific conditions that the extension must satisfy. These constraints reduce the space of preorders that are total extensions of the initial relations. As in the general case, we use the four steps to propose a proof by construction.

The paper is structured as follows: Section 2 recalls the formalisation of preferences, binary relations and properties used in the paper. Section 3 describes the proposed definition of compatibility and establishes a necessary and sufficient condition, using a modified version of Suzumura's definition of consistency. Section 4 presents the four proposed steps to establish an alternative and constructive proof for the sufficiency of the condition. Section 5 considers additional constraints to compatibility and shows that the four steps can be used to determine which conditions need to be added to satisfy these constraints.

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2 Preferences, Binary Relations and Properties

Preferences have been studied and formalised logically since the 20th century by philosophers, economists and mathematicians. This section recalls the usual formalism associated with binary relations and presents the consistency property proposed by Suzumura [11].

2.1 *Background*

Let $\mathbb{D}$ be a non-empty and finite set of decisions and R a binary relation on $\mathbb{D}$. R can be seen as a subset of $\mathbb{D}^2$: in the following, the notations $(x, y) \in R$ and xRy are used interchangeably. As is usual when handling preferences, the *symmetric part* of R is denoted $I(R)$ for *indifference*, even if we use in the paper the equivalent term of *equivalence*. The *asymmetric part* of R is denoted $P(R)$ for *preference*. It holds that $R = I(R) \cup P(R)$.

$$I(R) = \{(x, y) \in \mathbb{D}^2 \mid (x, y) \in R \wedge (y, x) \in R\}$$
$$P(R) = \{(x, y) \in \mathbb{D}^2 \mid (x, y) \in R \wedge (y, x) \notin R\}$$

Therefore a single binary relation R is sufficient to handle both equivalence and preference. A relation is said to be *symmetric* (resp. *asymmetric*), if its asymmetric (resp. symmetric) part is empty. Other common properties include, for all x, y, z in $\mathbb{D}^2$:

$$xRx \qquad (\textit{reflexive})$$
$$xRy \wedge yRz \Rightarrow xRz \qquad (\textit{transitive})$$
$$x \neq y \Rightarrow xRy \vee yRx \qquad (\textit{connected})$$

When a relation is both reflexive and connected, it is said to be *complete*. A *strict order* is a binary relation that is asymmetric and transitive. A *preorder* is a binary relation that is reflexive and transitive. It can be understood as an order that allows ties between decisions. It is therefore the most used binary relation to represent preferences. A preorder is often denoted $\succsim$, where $\sim$ and $\succ$ are respectively its symmetric and asymmetric parts. If an order or a preorder is connected, it is qualified as *total*, and if not as *partial*.

A *path* in a binary relation R is denoted xR^+y. There is a path from x to y if either xRy or $\exists m \geq 1,\ \exists a_1, \dots, a_m \in \mathbb{D}$ such that $xRa_1R \dots Ra_mRy$. A binary relation is *acyclic* if there is no path from a decision to itself. Besides, the inverse of a relation is noted $R^{-1} = \{(y, x) \in \mathbb{D}^2 \mid (x, y) \in R\}$.

Three more concepts are used in the paper. Firstly, a relation R admits a *PR-cycle* if $\exists x, y \in \mathbb{D}$ such that $xP(R)yR^+x$. In other words, it contains a cycle involving at least one asymmetrical comparison.

Secondly, the extension of a binary relation is defined as follows: τ is an *extension* of R if $R \subseteq \tau$ and $P(R) \subseteq P(\tau)$. As $R = I(R) \cup P(R)$ we also get $I(R) \subseteq I(\tau)$. Note that this definition of extensions considers that the symmetric and asymmetric parts must be preserved. This property is necessary when manipulating preferences, as these two parts have a different meaning. $Ext(R)$ denotes the set of extensions of R and $TExt(R)$ the set of *total extensions*, i.e. extensions that are total preorders.

Thirdly, the *transive closure* of a binary relation, denoted $tCl(R)$, is the smallest relation on $\mathbb{D}$ that contains R and is transitive. It is therefore contained in all transitive relations S that contain R: $R \subseteq S \Rightarrow tCl(R) \subseteq S$. All binary relations have a transitive closure. It is useful to note that $(x, y) \in tCl(R) \Leftrightarrow xR^+y$.

2.2 *Suzumura's Consistency*

The main contribution of this paper is based on a property proposed by Suzumura [11] that we denote *S-consistency*, since different properties are also referred to as consistency by other authors. S-consistency is based on the absence of PR-cycles of any order.

Definition 1 (S-consistency)**.**
R is S-consistent iff R contains no PR-cycles.

This property is used in a variety of contexts as discussed in more details in [1, 7, 2]. It is weaker than transitivity, *transitivity* $\Rightarrow$ *S-consistency*, while being stronger than acyclicity of the asymetric part. It has also been shown that, in case of completeness, S-consistency is equivalent to transitivity. It was introduced by Suzumura as a condition for the existence of an extension that is a total preorder.

Theorem 1 (Suzumura, Theorem 3 in [11])**.**
R is S-consistent $\Leftrightarrow TExt(R) \neq \emptyset$.

This theorem answers the question whether a given relation has a total extension. It lies in a group of studies that propose conditions for the existence of an extension. They are all based on Szpilrajn's fundamental theorem that establishes that all partial orders can be extended to a total order [12]. Hansson [5] proves that partial preorders all have a total extension. These two proofs are detailed and thus self-contained in Fishburn [4] and then used by Suzumura to prove the theorem. Therefore, as it establishes an equivalence, S-consistency is the weakest condition to the existence of an extension.

3 Proposed Definition for Preorder Compatibility

Let us consider n preorders $R_1, \dots, R_n \in \mathbb{D}^2$ that correspond to preferences from different sources. We propose a definition of compatibility and a necessary and sufficient condition for it to hold. This condition is a modified version of S-consistency that applies to the set of preorder $R_1, \dots, R_n$. The definition of compatibility is:

Definition 2 (Compatibility between preorders)**.**
Preorders $R_1, \dots, R_n$ are compatible iff $\bigcap_{i=1}^{n} TExt(R_i) \neq \emptyset$.

This means that $R_1, \dots, R_n$ are compatible if and only if there exists at least one extension that is shared by all preorders and is a total preorder. As the extension is shared, compatibility involves the ability to combine the preorders in a single relation that satisfies the properties desirable for preferences. As discussed below, it implies that for a given $\mathbb{D}$, the union is a satisfactory aggregator on $R_1, \dots, R_n$. Conversely, when the preorders are not compatible, it shows that a more complex aggregation procedure is needed to produce satisfactory preferences. In the case of ethical principles, this may involve restricting their application or simply rejecting them as unsatisfactory.

In the rest of this section, we propose a necessary and sufficient condition for compatibility, so as to avoid having to generate the set of extensions for each preorder to check compatibility. Doing so also provides some insight on the reason why the union is a satisfactory aggregator when the preorders are compatible. We show that the proposed definition of compatibility is related to S-consistency. For the sake of legibility, let us denote $\bigcup_r = \bigcup_{i=1}^n R_i$, $\bigcup_{pr} = \bigcup_{i=1}^n P(R_i)$ and $\bigcup_{ir} = \bigcup_{i=1}^n I(R_i)$.

Theorem 2. *The preorders $R_1, \dots, R_n$ are compatible*

$$\Leftrightarrow \begin{cases} (A)\ \forall x, y \in \mathbb{D}, \forall j, k, (x, y) \in P(R_j) \Rightarrow (y, x) \notin R_k \\ (B)\ TExt(\bigcup_r) \neq \emptyset \end{cases}$$

To prove this theorem, we use the following lemma

Lemma 3. $(A) \Leftrightarrow \bigcup_{pr} \subseteq P(\bigcup_r)$

Proof. $\Rightarrow$ Let us choose $(x, y) \in \bigcup_{pr}$, thus $\exists j \in [\![1, n]\!]$ such that $(x, y) \in P(R_j)$. We know that $\forall k \in [\![1, n]\!], (y, x) \notin R_k$, thus $(y, x) \notin \bigcup_r$. Now $(x, y) \in \bigcup_r$ and thus $(x, y) \in P(\bigcup_r)$

$\Leftarrow$ Let us choose $(x, y) \in P(R_j)$. We have $(x, y) \in \bigcup_{pr}$ and so we deduce $(x, y) \in P(\bigcup_r)$. Thus, $(y, x) \notin \bigcup_r$ □

Proof of Theorem 2. Necessity $\Rightarrow$: Let us suppose that $\exists \tau \in \mathbb{D}^2$, $\tau \in \bigcap_{i=1}^n TExt(R_i)$, which means that $\forall i \in [\![1, n]\!], R_i \subseteq \tau \wedge P(R_i) \subseteq P(\tau)$.

Let us prove (A) using Lemma 3. We consider $(x, y) \in \mathbb{D}^2$ such that $(x, y) \in \bigcup_{pr}$. Thus $(x, y) \in \bigcup_r$. It is then sufficient to show that $(y, x) \notin \bigcup_r$. Let us choose $j \in [\![1, n]\!]$ such that $(x, y) \in P(R_j)$. Knowing that $P(R_j) \subseteq P(\tau)$, we get $(x, y) \in P(\tau)$, which gives $(y, x) \notin \tau$. As $R_i \subseteq \tau$, we get $(y, x) \notin \bigcup_r$.

Let us prove (B). Let us take $\tau \in \bigcap_{i=1}^n TExt(R_i)$ and prove that τ is an extension of $\bigcup_r$: we only have to prove $P(\bigcup_r) \subseteq P(\tau)$ since by definition of $\bigcap_{i=1}^n TExt(R_i)$ we know $\bigcup_r \subseteq \tau$. Let us choose $(x, y) \in P(\bigcup_r)$, thus $(x, y) \in \bigcup_r$ and $(y, x) \notin \bigcup_r$. Therefore $\exists R_j$ such that $(x, y) \in R_j$ and that $(x, y) \in P(R_j)$. Knowing that $\tau \in TExt(R_j)$, $(x, y) \in P(\tau)$. We have $P(\bigcup_r) \subseteq P(\tau)$ thus τ is an extension of $\bigcup_r$. Furthermore τ is a total preorder by definition of $TExt(R_i)$. Thus $\tau \in TExt(\bigcup_r)$.

Sufficiency $\Leftarrow$: Let us suppose (A) and (B). With (B), let us take $\sigma \in TExt(\bigcup_r)$, thus we have $\bigcup_r \subseteq \sigma$ and $\forall i \in [\![1, n]\!], R_i \subseteq \sigma$. We must prove that $\forall i \in [\![1, n]\!], P(R_i) \subseteq P(\sigma)$. Let us choose (x, y) and R_j such that $(x, y) \in P(R_j)$. Therefore, $(x, y) \in \bigcup_{pr}$. With (A) and Lemma 3, we get $(x, y) \in P(\bigcup_r)$. Knowing by (B) that $P(\bigcup_r) \subseteq P(\sigma)$, we get $(x, y) \in P(\sigma)$. □

This theorem makes explicit the link between compatibility and the fact that the union is a satisfactory aggregator. Indeed, the two conditions (A) and (B) identify the two possible cases of incompatibility for the union. (A) is used to generalise the asymmetry to a group of relations. This prohibits all PR-cycles of size two between relations. On the other hand (B) is equivalent to S-consistency and ensures that there are no PR-cycles of size strictly greater than two between the relations. Indeed by decomposing the union into its symmetrical and asymmetrical part, S-consistency cannot detect cases of cycles of size two as they are then considered as equivalence. It is easily shown by considering xR_1y and yR_2x. Indeed, in that case $(x, y), (y, x) \in \bigcup_{pr}$ but $(x, y), (y, x) \in I(\bigcup_r)$. Finally, using this theorem, we prove that the compatibility we propose is equivalent to a slightly modified version of the S-consistency property for a set of binary relations, denoted (Q).

Proposition 4. *The preorders* $R_1, ..., R_n$ *are compatible* $\Leftrightarrow$
$$\forall x, y \in \mathbb{D},\ x \textstyle\bigcup_r^+ y \Rightarrow (y, x) \notin \bigcup_{pr} \qquad (Q)$$

Proof. $\Rightarrow$ By Theorem 2 we know (B) and with Theorem 1 that $\bigcup_r$ admits no PR-cycle. It implies that $\forall x, y \in \mathbb{D},\ x \bigcup_r^+ y \Rightarrow (y, x) \notin P(\bigcup_r)$. With (A) and Lemma 3, we get $\bigcup_{pr} \subseteq P(\bigcup_r)$ and thus $\forall x, y \in \mathbb{D},\ x \bigcup_r^+ y \Rightarrow (y, x) \notin \bigcup_{pr}$.

$\Leftarrow$ If $x \bigcup_r y$ this implication is a rewriting of the contraposition of (A). Knowing (A) and Lemma 3, we get $(x_1, x_2) \in P(\bigcup_r)$. Thus $\bigcup_r$ does not admit PR-cycles, and with Theorem 1, we deduce (B). Finally, with Theorem 2, we get that $R_1, ..., R_n$ are compatible. □

The proposed modification of S-consistency lies only in considering that the asymmetric comparison must be in an asymmetric part of one R_i, $\bigcup_{pr} = \bigcup_{i=1}^n P(R_i)$, rather than on the asymmetric part of the union $P(\bigcup_r) = P(\bigcup_{i=1}^n R_i)$.

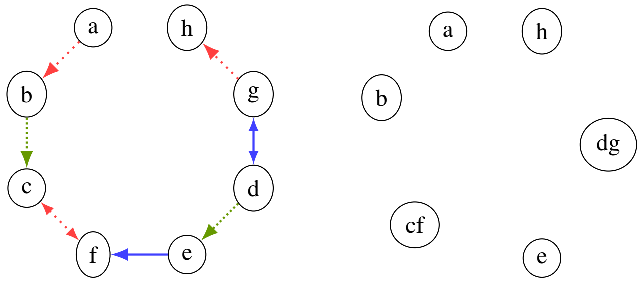

Figure 1. (Left) Three considered preorders R_1 (red dotted arrows), R_2 (green densely dotted arrow) and R_3 (blue full arrows). (Right) necessary equivalence classes

4 Constructive Proof without Suzumura's theorem

All the proofs about the existence of extensions presented in the previous sections consider an infinite set of decisions and thus use theoretical mathematical tools to handle infinity (see [8], pages 31-36). We propose here an alternative and constructive proof. To construct a compatible preorder, it assumes that the set of decisions is finite, which represents a significant part of the decision-making problems, especially those considered in computational ethics. The sufficiency of the condition is proved by constructing a total preorder that belongs to $\bigcap_{i=1}^n TExt(R_i)$. In the following four subsections, we describe in turn four steps used to construct this preorder. The first two steps determine the equivalences and preferences that are necessarily required to satisfy the transitivity of a compatible preorder. In the third step, these comparisons are combined to build a preorder, which is then extended into a total preorder in the final step. One particular extension is proposed for an ethical context. The steps are illustrated considering the example shown on the left part of Figure 1: for the set of decisions $\mathbb{D} = \{a, b, c, d, e, f, g, h\}$, three preorders R_1, R_2 and R_3 are considered.

4.1 Determining Necessary Equivalences

The aim of the first step is to determine the set of equivalences, denoted $N_\sim$, which are necessarily required in the extension. To show that they are necessary, we need to prove that the compatibility of $R_1, \ldots, R_n$ implies $\forall \tau \in \bigcap_{i=1}^n TExt(R_i),\ N_\sim \subseteq I(\tau)$. In this step and the next one only, we therefore assume that the preorders $R_1, \ldots, R_n$ are compatible and we note τ an element of $\bigcap_{i=1}^n TExt(R_i)$.

Proposition 5. $\tau \in \bigcap_{i=1}^n TExt(R_i) \Rightarrow tCl\left(\bigcup_{ir}\right) \subseteq I(\tau)$

Proof. By definition of $\bigcap_{i=1}^n TExt(R_i)$, we know that τ is a extension of all R_i, and thus $\bigcup_{ir} \subseteq I(\tau)$. Knowing that τ is a total preorder, $I(\tau)$ is reflexive, symmetric and transitive. The closure being minimal, $tCl\left(\bigcup_{ir}\right) \subseteq I(\tau)$. □

Thus we note $N_\sim = tCl\left(\bigcup_{ir}\right)$. The union and the transitive closure preserve the reflexivity and symmetry of $I(R_i)$, so $N_\sim$ is an equivalence relation on $\mathbb{D}$. Therefore it decomposes $\mathbb{D}$ into equivalence classes. The equivalence classes for the illustrative example with R_1, R_2, R_3 are represented in the right part of Figure 1.

4.2 Determining Necessary Preferences

In the same spirit, the set of preferences that are necessarily present in the extension is denoted $N_\succ$. The aim is to prove that the compatibility of $R_1, \ldots, R_n$ implies $\forall \tau \in \bigcap_{i=1}^n TExt(R_i),\ N_\succ \subseteq P(\tau)$.

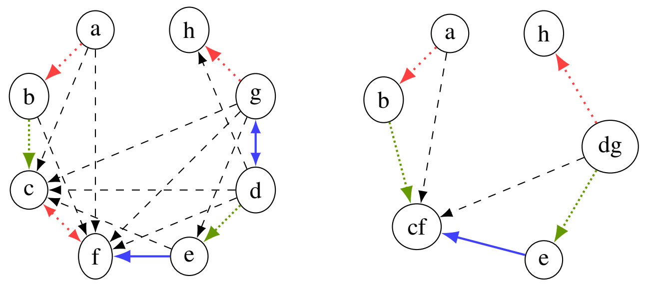

Figure 2. (Left) necessary preferences, where black and dashed arrows represent the added preferences. (Right) $\succ_{\mathbb{C}}$, projection of the necessary preferences on the equivalence classes.

Proposition 6. $\tau \in \bigcap_{i=1}^{n} TExt(R_i) \Rightarrow$
$\forall x, y \in \mathbb{D}, \left[\exists a, b \in \mathbb{D},\ x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ y\right] \Rightarrow (x, y) \in P(\tau)$

Proof. Assume $\exists a, b \in \mathbb{D}$, $x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ y$. Knowing $\tau \in \bigcap_{i=1}^{n} TExt(R_i)$, which means $\bigcup_r \subseteq \tau$ and $\bigcup_{pr} \subseteq P(\tau)$, we get $x\, \tau^+ a P(\tau) b\, \tau^+ y$. By transitivity, we get $(x, y) \in \tau$. Let us suppose ad absurdum that $(y, x) \in \tau$, then $b\ \tau^+ a$ and by transitivity $(b, a) \in \tau$. As $(a, b) \in P(\tau)$, it is absurd. □

We note $N_\succ = \{(x, y) \in \mathbb{D}^2 \mid \exists a, b \in \mathbb{D},\ x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ y\}$. For the considered illustrative example, the necessary preferences that are added to R_1, R_2, R_3 by the closure are represented by the black and dashed arrows in Figure 2. As each $R_1, \ldots, R_n$ is reflexive, so is $\bigcup_r$. Therefore for all (x, y) in $\bigcup_{pr}$ it holds that $x \bigcup_r x \bigcup_{pr} y \bigcup_r y$ and thus (x, y) is in $N_\succ$. This establishes that $\bigcup_{pr} \subseteq N_\succ$. Moreover, $N_\succ$ is transitive and asymmetric under the considered condition.

Proposition 7. *$N_\succ$ is transitive and $[(Q) \Rightarrow N_\succ$ asymmetric$]$.*

Proof. Transitivity: let us suppose that $\exists x, y, z \in \mathbb{D}$ such that $x N_\succ y$ and $y N_\succ z$. By definition of $N_\succ$, $\exists a, b, c, d \in \mathbb{D}$ such that $x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ y \bigcup_r^+ c \bigcup_{pr} d \bigcup_r^+ z$. Knowing that $\bigcup_{pr} \subseteq \bigcup_r$, $a \bigcup_r b$, thus $x \bigcup_r^+ c \bigcup_{pr} d \bigcup_r^+ z$, therefore $(x, z) \in N_\succ$.

Asymmetry: let us suppose that $\exists x, y \in \mathbb{D}$ such that $(x, y) \in N_\succ$. Ad absurdum, assume that $(y, x) \in N_\succ$. By transitivity of $N_\succ$, it holds that $(x, x) \in N_\succ$. By definition of $N_\succ$, $\exists a, b \in \mathbb{D}$ such that $x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ x$. Therefore we get $b \bigcup_r^+ a$. Using (Q), it holds that $(a, b) \notin \bigcup_{pr}$, which is absurd. □

4.3 *Building a Preorder*

The goal of this step is to use the necessary comparisons defined previously in order to build a preorder that is an extension of all $R_1, \ldots, R_n$, i.e. that is an element of $\bigcap_{i=1}^{n} Ext(R_i)$. The following theorem makes explicit all the conditions that an equivalence relation and a strict order must satisfy so that their union is such an extension.

Theorem 8. *Let us consider $\succ, \sim \subseteq \mathbb{D}^2$ such that $\succ$ is a strict order and $\sim$ is an equivalence relation. Then it holds that:*

$$\begin{cases} \bigcup_{pr} \subseteq \succ \\ \bigcup_{ir} \subseteq \sim \\ \succ \cap \sim = \emptyset \end{cases} \Rightarrow (\succ \cup \sim) \in \bigcap_{i=1}^{n} Ext(R_i)$$

Proof. Let us take $i \in [\![1, n]\!]$. We have to prove that $R_i \subseteq (\succ \cup \sim)$ and $P(R_i) \subseteq P(\succ \cup \sim)$. As it is assumed that $\bigcup_{pr} \subseteq \succ$ and $\bigcup_{ir} \subseteq \sim$, it holds that $P(R_i) \subseteq \succ$ and $I(R_i) \subseteq \sim$. Thus $I(R_i) \cup P(R_i) \subseteq (\succ \cup \sim)$: $R_i \subseteq (\succ \cup \sim)$ holds. By union, $P(R_i) \subseteq \bigcup_{pr}$, let us prove $P(\succ \cup \sim) = \succ$. We know that $\succ$ is asymmetric and that $\sim$ is symmetric. Furthermore, as $\succ \cap \sim = \emptyset$, for each $(x, y) \in \succ$, we know that $(y, x) \notin (\succ \cup \sim)$. Thus $P(\succ \cup \sim) = \succ$. We get $P(R_i) \subseteq \bigcup_{pr} \subseteq \succ = P(\succ \cup \sim)$.

As such $(\succ \cup \sim)$ is an extension of R_i and as the same reasoning applies for any $i \in [\![1, n]\!]$, it holds that $\succsim \in \bigcap_{i=1}^{n} Ext(R_i)$. □

As we want to apply this theorem to the necessary comparisons introduced in the previous sections, the following proposition establishes that $N_\sim \cap N_\succ = \emptyset$. Indeed it has already been shown that $N_\sim$ is an equivalence relation and by definition of the transitive closure, it holds that $\bigcup_{ir} \subseteq N_\sim$. Similarly for $N_\succ$, Proposition 7 establishes that it is a strict order and $\bigcup_{pr} \subseteq \succ$.

Proposition 9. $(Q) \Rightarrow N_\sim \cap N_\succ = \emptyset$

Proof. Assume, ad absurdum, $(x, y) \in N_\sim$ and $(x, y) \in N_\succ$. By definition of $N_\succ$, $\exists a, b \in \mathbb{D}$, such that $x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ y$. Moreover, we get $(y, x) \in N_\sim$ by symmetry of $N_\sim$, thus by definition of the transitive closure $y \bigcup_{ir}^+ x$. Knowing that $\bigcup_{ir} \subseteq \bigcup_r$, it holds that $y \bigcup_r^+ x$. Therefore, we get $b \bigcup_r^+ a$. By applying (Q) we get $(a, b) \notin \bigcup_{pr}$, which is absurd. □

All the conditions of Theorem 8 are satisfied by $N_\sim$ and $N_\succ$. Thus it establishes that $(N_\sim \cup N_\succ) \in \bigcap_{i=1}^{n} Ext(R_i)$. Let us note $\succsim = N_\sim \cup N_\succ$. Therefore $\succsim$ is an extension of all $R_1, \ldots, R_n$. To conclude this step, let us prove that $\succsim$ is a preorder.

Proposition 10. *$\succsim$ is transitive and reflexive.*

Proof. As $N_\sim$ is reflexive, so is $\succsim$. To prove that it is also transitive, let us assume that $\exists x, y, z \in \mathbb{D}$ such that $(x, y), (y, z) \in \succsim$. Four cases are distinguished:

- $(x, y), (y, z) \in N_\sim$: Then by transitivity of $N_\sim$ it holds that $(x, z) \in N_\sim$ and thus $(x, z) \in \succsim$.
- $(x, y), (y, z) \in N_\succ$: Then by transitivity of $N_\succ$ it holds that $(x, z) \in N_\succ$ and thus $(x, z) \in \succsim$.
- $(x, y) \in N_\sim$ and $(y, z) \in N_\succ$: As $N_\sim$ is a transitive closure, it holds that $x \bigcup_{ir}^+ y$ and thus, as $\bigcup_{ir} \subseteq \bigcup_r$, $x \bigcup_r^+ y$. Moreover, by definition of $N_\succ$, it holds that $\exists a, b \in \mathbb{D}$, $y \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ z$. Linking the two, we get $x \bigcup_r^+ a \bigcup_{pr} b \bigcup_r^+ z$. By definition of $N_\succ$, it gives us $(x, z) \in N_\succ$ and thus $(x, z) \in \succsim$.
- $(x, y) \in N_\succ$ and $(y, z) \in N_\sim$: the reasoning of the previous case give us $(x, z) \in \succsim$.

In all cases, it holds that $(x, z) \in \succsim$, therefore $\succsim$ is transitive. □

4.4 *Extending the Preorder*

Having demonstrated that $\succsim$ is a preorder and an extension of $R_1, \ldots, R_n$, the final step extends it into a total preorder. To that aim, we first exploit a topological sort [9] to provide a proof by construction of Theorem 11 of Hansson (Lemma 3 in [5]). Then we argue that the longest path algorithm [3, 6] provides an alternative construction which is more relevant in an ethical context.

Theorem 11 (Hansson, 1968)**.** *Let R be a preorder, $TExt(R) \neq \emptyset$.*

$I(R)$ is an equivalence relation on $\mathbb{D}$. It decomposes $\mathbb{D}$ into a finite number of equivalence classes, the set of which is denoted $\mathbb{C}$. To simplify the notation, we consider the projection of $P(R)$ on $\mathbb{C}$:

$$\succ_{\mathbb{C}} = \{(C_1, C_2) \in \mathbb{C}^2 \mid \exists x, y \in \mathbb{D}, x \in C_1, y \in C_2, x P(R) y\}$$

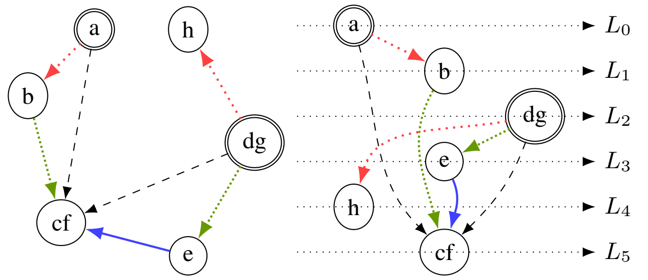

Figure 3. (Left) identification of maximal equivalence classes, shown as circled nodes; (Right) an ordering using a topological sorting algorithm

This projection is illustrated in Figure 2. Finding a total extension of R is equivalent to finding a layering of the directed graph associated with $\succ_{\mathbb{C}}$. This involves the assignment of all the equivalence classes to a layer. The layers are denoted $L_0, \ldots, L_l$. On a given layer, all decisions are considered equivalent and those assigned to a higher layer (i.e. with greater index) are regarded as less preferable. Layer L_0 represents the preferred decisions and L_l the least preferred ones. For a layering to satisfy $\succ_{\mathbb{C}}$, it must verify that $\forall (C_a, C_b) \in \succ_{\mathbb{C}}, [C_a \in L_c \wedge C_b \in L_d \Rightarrow c < d]$.

To build such a layering, $\succ_{\mathbb{C}}$ must be acyclic. It is proved in the following proposition:

Proposition 12. $\succ_{\mathbb{C}}$ *is acyclic*

Proof. Assume, ad absurdum, $\exists C_1, \ldots, C_m \in \mathbb{C}$ such that $C_1 \succ_{\mathbb{C}} \cdots \succ_{\mathbb{C}} C_m \succ_{\mathbb{C}} C_1$. Then $\exists a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{D}$ such that $(\forall a_i, b_i \in C_i)$, $(\forall i \in [\![1, m-1]\!], \; a_i P(R) b_{i+1})$ and $(a_m P(R) b_1)$. Knowing that $a_i, b_i \in C_i$, it holds that $b_i I(R) a_i$, thus $b_i R a_i$. Moreover, as $P(R) \subseteq R$, it implies $b_1 R a_1 R b_2 R \ldots R b_m R a_m$, i.e. $b_1 R^+ a_m$ and by transitivity of R we get $b_1 R a_m$. This is absurd as $a_m P(R) b_1$. □

Once acyclicity is verified, a topological sort [9] of the directed graph can be performed. This layering approach assumes that all layers are composed of a single equivalence class. It represents a total preorder on $\mathbb{D}$ that is an extension of R, thus it proves Theorem 11.

However, we argue that this construction is not satisfying in an ethical context. To explain why, let us take a closer look at the set of equivalence classes that are not dominated for $\succ_{\mathbb{C}}$: $max_{\succ_{\mathbb{C}}} = \{C \in \mathbb{C} \mid \forall C' \in \mathbb{C}, (C', C) \notin \succ_{\mathbb{C}}\}$. The elements of $max_{\succ_{\mathbb{C}}}$ are called *maximal equivalence classes*. They are identified in Figure 3 with circled nodes. The topological ordering depicted in Figure 3 introduces preferences that are not justified ethically: decisions a and b are preferred to d and g. We argue that the introduction of arbitrary preferences among maximal equivalence classes is problematic if the preorder is used in a ethical decision analysis context, as it can favour decisions on an unethical basis.

Thus, in the remainder of this section, we put forth an ethical interpretation of an alternative layering, induced by the longest path algorithm [3, 6]. As represented in Figure 4, each decision is assigned to the layer whose index is the size of the longest path to a decision belonging to a maximal equivalence class. Denoting $l(x, y)$ the length of the path between x and y in the built directed graph $\succ_{\mathbb{C}}$, the size of the longest path from a decision x to a decision belonging to a maximal equivalence class is formally defined as

$$d_{\succ_{\mathbb{C}}}(x) = \begin{cases} 0 & \text{if } x \in C_x \wedge C_x \in max_{\succ_{\mathbb{C}}} \\ \max\{l_{path}(x, y) | y \in C_y, C_y \in max_{\succ_{\mathbb{C}}}\} & \text{otherwise} \end{cases}$$

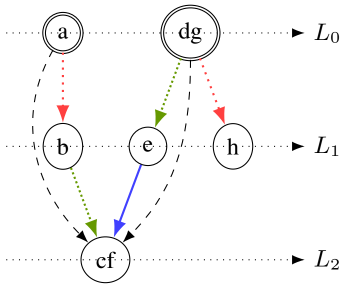

Figure 4. The ordering induced by the longest path layering algorithm

The highest layer is labelled with $l = \max_x d_{\succ_{\mathbb{C}}}(x)$, the size of the longest path within $\succ_{\mathbb{C}}$. We assign to each decision one layer of $L_0, \ldots, L_l \subseteq \mathbb{D}$ according to the size of their longest path to a maximal equivalence class: $L_i = \{x \in \mathbb{D} \mid d_{\succ_{\mathbb{C}}}(x) = i\}$. These layers form a total extension of the strict order $\succ_{\mathbb{C}}$ and therefore a total preorder of $\mathbb{D}$, denoted $R^\star$:

$$R^\star = \{(x, y) \in \mathbb{D} \mid x \in L_i \wedge y \in L_j \wedge i \leq j\}$$

As with a topological sort, the longest path algorithm enables us to prove the following proposition, which is also a proof of Theorem 11:

Proposition 13. $R^\star$ *is a total preorder and* $R^\star \in Ext(R)$

This layering always locates the maximal equivalence classes in the preferred layer L_0: it introduces equivalence comparisons between them. Thus, it preserves the fundamental characteristic of these decisions: they are not dominated. By doing so, any decision-making process is less susceptible to be influenced by unethically justified preferences, among this preferred decisions.

This layering also provides an interesting information for the decisions which are not in a maximal equivalence class. Indeed, the layer of the decision indicates the extent to which the agent has deviated from an optimal ethical decision. For instance, decision e is in layer L_2: according to all the considered principles, one could have performed a better decision twice. Therefore as depicted in Figure 4, the layering is an effective representation and thus gives a clear understanding of the ethical relations between the decisions in case of compatibility.

One must note that this layering introduces new preferences: h is e.g. preferred to c and f. However, it never adds them among the maximal equivalence classes and these preferences only indicate that h is less far away from a optimal ethical decision than c or f. As such it must not be conflated with a genuine ethical preference inferred by a principle. Indeed, h could actually be worse than c, for other ethical reasons, not included in the currently considered principles. This layering is only inferred from the considered principles, therefore it will provide more satisfying information if these principles encompass all the ethical components of the given problem.

5 Adding More Constraints to $R_1, \ldots, R_n$

In the previous sections, we proposed a definition of compatibility for a set of preference relations $R_1, \ldots, R_n$ as well as four steps to construct an extension. In this section, we show that if one wants to add constraints in addition to the preferences, the four proposed steps are applicable and facilitate the determination of new conditions for compatibility. To illustrate this, we define two constraints before applying the steps.

5.1 Adding Constraints

As constraints, let us consider other types of comparisons that can be qualified as *unsure*. We avoid the term uncertainty as it is more scientifically connoted. The constraints we consider are meant to illustrate the efficiency of the steps and not to propose a precise formalisation of ethical principles under uncertainty. The first unsure comparison defines the *weak preferences*, *at least as good as*, and is denoted $W \in \mathbb{D}^2$. This relation is unsure as it does not distinguish strict preferences from equivalences. W must be distinguished from any $R_1, \ldots, R_n$ which are only relations that allow to manipulate well-defined preferences and equivalences together: $R_i = P(R_i) \cup I(R_i)$. The second unsure comparison is a *difference* relation, denoted $D \in \mathbb{D}^2$, which, in a similar way to W, is not able to distinguish a strict preference from its inverse. If we consider that there are only three comparisons between objects, namely better than, equivalent to and worse than, these unsure comparisons can be seen as negative constraints: xWy means that y is not better than x and xDy that x is not equivalent to y.

To propose a formal definition of compatibility with these new constraints, let us first define them:

Definition 3 (Satisfiability of W)**.**
W is satisfied by $\tau \in \mathbb{D}^2$, iff $W \subseteq \tau \, (= I(\tau) \cup P(\tau))$

Definition 4 (Satisfiability of D)**.**
D is satisfied by $\tau \in \mathbb{D}^2$, iff $D \subseteq P(\tau) \cup P(\tau)^{-1}$.

Definition 5 (Compatibility with constraints)**.**
$R_1, \ldots, R_n$, W and D are compatible $\Leftrightarrow \exists \tau \in \bigcap_{i=1}^{n} TExt(R_i)$ such that W and D are satisfied by τ.

5.2 Determining Necessary Equivalences

As before, we assume that the preorders $R_1, \ldots, R_n, W$ and D are compatible, thus we note τ an element of $\bigcap_{i=1}^{n} TExt(R_i)$ that satisfies W and D. W is going to extend the set of necessary equivalence but not D, as it can be seen as a negation of equivalence. To improve legibility, let us note $T = \bigcup_{ir} \cup W$ and $\mathcal{C}_\sim = \{(x, y) \in \mathbb{D}^2 \mid xT^+yT^+x\}$ the set of pairs of $\mathbb{D}$ that are in a cycle of T.

Proposition 14.
$R_1, \ldots, R_n$, W and D are compatible $\Rightarrow \mathcal{C}_\sim \subseteq I(\tau)$.

Proof. As compatibility is assumed, $\exists \tau \in \bigcap_{i=1}^{n} TExt(R_i)$ such that W, D are satisfied by τ. Let us consider a cycle $(xTa_1T \ldots Ta_mTyTb_1T \ldots Tb_pTx)$ and two decisions in this cycle, z_i and z_j. Knowing that τ satisfies W: $W \subseteq \tau$. Moreover, by definition of $\bigcap_{i=1}^{n} TExt(R_i), \forall i \in [\![1, n]\!]$, $I(R_i) \subseteq \tau$. Thus this is also a cycle on τ. By transitivity of τ, it holds that both $z_i \tau z_j$ and $z_j \tau z_i$. Thus, $(z_i, z_j) \in I(\tau)$. ☐

Let us denote $N'_\sim$ the transitive closure of these necessary equivalences: $N'_\sim = tCl\left(\bigcup_{ir} \cup \mathcal{C}_\sim\right)$. As $I(\tau)$ is transitive by definition and $\left(\bigcup_{ir} \cup \mathcal{C}_\sim\right) \subseteq I(\tau)$ and as the transitive closure is minimal, we get $N'_\sim \subseteq I(\tau)$. Moreover, the union and the transitive closure preserve the reflexivity and symmetry of $I(R_i)$ and $\mathcal{C}_\sim$, so $N_\sim$ is an equivalence relation on $\mathbb{D}$.

5.3 Determining Necessary Preferences

Necessary preferences are slightly more complex as both W and D add necessary preferences. For legibility, let us note $U = \bigcup_r \cup W$ and $U_{pr} = \bigcup_{pr} \cup (W \cap D)$. We then define $N'_\succ = \{(x, y) \in \mathbb{D}^2 \mid \exists a, b \in \mathbb{D},\ xU^+aU_{pr}bU^+y\}$.

Proposition 15.
$R_1, \ldots, R_n$, W and D are compatible $\Rightarrow N'_\succ \subseteq P(\tau)$

Proof. Assume $\exists x, y, a, b \in \mathbb{D}, \quad xU^+aU_{pr}bU^+y$. Knowing that $R_1, \ldots, R_n$, W and D are compatible, it holds that $\tau \in \bigcap_{i=1}^{n} TExt(R_i)$, which means that $\bigcup_r \subseteq \tau$ and $\bigcup_{pr} \subseteq P(\tau)$. As τ satisfies W and D, it holds that $W \subseteq \tau$. Moreover, $W \cap D \subseteq \tau \cap \left(P(\tau) \cup P(\tau)^{-1}\right)$. By definition of the inverse of an asymmetric part $\tau \cap P(\tau)^{-1} = \emptyset$, thus we get $W \cap D \subseteq P(\tau)$. Therefore $U \subseteq \tau$ and $U_{pr} \subseteq P(\tau)$. We get $x\,\tau^+aP(\tau)b\,\tau^+y$. By transitivity, we get $(x, y) \in \tau$. Ad absurdum, let us suppose that $(y, x) \in \tau$, therefore $b\ \tau^+a$ and by transitivity $(b, a) \in \tau$, which is absurd knowing $(a, b) \in P(\tau)$. This proves $(x, y) \in P(\tau)$. ☐

Proposition 16. *$N'_\succ$ is transitive*

Proof. This proposition is proved as Proposition 7. ☐

5.4 Building a Preorder

To be able to apply Theorem 8, two prerequisites are missing: the asymmetry of $N'_\succ$ and the fact that $N'_\sim \cap N'_\succ = \emptyset$. Both are therefore defined as conditions in order to prove that $R_1, \ldots, R_n$, W and D are compatible. Thus using Theorem 8 and by denoting $\succsim' = N'_\sim \cup N'_\succ$ it holds that $\succsim' \in \bigcap_{i=1}^{n} Ext(R_i)$.

Proposition 17. *$\succsim'$ is transitive and reflexive.*

Proof. As $N'_\sim$ is reflexive, so is $\succsim'$. To prove that it is also transitive, let us assume that $\exists x, y, z \in \mathbb{D}$ such that $(x, y), (y, z) \in \succsim'$. Four cases are distinguished:

- $(x, y), (y, z) \in N'_\sim$: Then by transitivity of $N'_\sim$ it holds that $(x, z) \in N'_\sim$ and thus $(x, z) \in \succsim'$.
- $(x, y), (y, z) \in N'_\succ$: we get $(x, z) \in \succsim'$ with the same reasoning.
- $(x, y) \in N_\sim$ and $(y, z) \in N_\succ$: By definition of the closure, it holds that $x(\bigcup_{ir} \cup \mathcal{C}_\sim)^+y$. Knowing that $a\mathcal{C}^+_\sim b$ means that aT^+bT^+a, with $T = \bigcup_{ir} \cup W$. Knowing that $U = \bigcup_r \cup W$ and $\bigcup_{ir} \subseteq \bigcup_r$, it holds that $\mathcal{C}_\sim \subseteq U$. Thus, $(\bigcup_{ir} \cup \mathcal{C}_\sim) \subseteq U$ it holds that xU^+y. Moreover, by definition of $N'_\succ$, it holds that $\exists a, b \in \mathbb{D}, \quad yU^+aU_{pr}bU^+z$. Linking the two, we get $xU^+aU_{pr}bU^+z$. By definition of $N'_\succ$, it gives us $(x, z) \in N'_\succ$ and thus $(x, z) \in \succsim'$.
- $(x, y) \in N'_\succ$ and $(y, z) \in N'_\sim$: the reasoning of the previous case give us $(x, z) \in \succsim'$.

In all cases, it holds that $(x, z) \in \succsim'$, therefore $\succsim'$ is transitive. ☐

5.5 Extending the Preorder

To extend the preorder, this section uses the layering algorithms introduced in Section 4. They must here ensure that W and D are satisfied. As this section intends to illustrate the efficiency of the steps when considering constraints, the layering introduces preferences instead of equivalences in order to satisfy W and D, which is a convenient choice, but not the only possible one.

As all cycles of W are in $N'_\sim$ by its construction, the part of W outside of the equivalence classes is acyclic. Besides, as $N'_\succ$ is supposed to be asymmetric, $N'_\succ \cup W$ is acyclic by construction of $N'_\succ$.

Therefore all equivalence classes can be layered according to $N'_\succ$, using the longest path algorithm. These layers are denoted $K_1, \ldots, K_d$. Then among each layer, a topological sort is applied to sort the equivalence classes to ensure the satisfiability of the constraints. The final layers correspond to an indexation of all classes, denoted $C_1, \dots, C_e$. Formally we have that $\forall k \in [\![1, d]\!], \forall a, b \in K_k$ such that $a \in C_i, b \in C_j$ and $i \neq j$, it holds that $(aWb \Rightarrow i < j)$. This construction enables us to propose the following extension:

$$\succsim'^{\star} = \{(x, y) \in \mathbb{D} \mid x \in K_i, C_j \wedge y \in K_k, C_l \wedge [(i < k) \vee (i = k \wedge j \leq l)]\}$$

Four properties must be proven for $\succsim'^{\star}$: $\succsim'^{\star}$ is a total preorder, $\succsim'^{\star} \in Ext(\succsim')$ (thus $\succsim'^{\star} \in \bigcap_{i=1}^{n} TExt(R_i)$), $\succsim'^{\star}$ satisfy W and $\succsim'^{\star}$ satisfy D.

Proposition 18. *$\succsim'^{\star}$ is a total preorder*

Proof. Let us prove that it is complete. Let us consider $(x, y) \in \mathbb{D}^2$. By the decomposition into equivalent classes, $\exists j, l \in [\![1, e]\!]$ such that $x \in C_j$ and $y \in C_l$. By construction of $K_1, \dots, K_d$, we get $\exists i, k \in [\![1, d]\!]$ such that $x \in K_i$ and $y \in K_k$. Therefore, either $i \neq k$ and if $i < k$ then $(x, y) \in \succsim'^{\star}$, else $k < i$ then $(y, x) \in \succsim'^{\star}$, or $i = k$. In that case, either $j \leq l$ and $(x, y) \in \succsim'^{\star}$ or $l \leq j$ and $(y, x) \in \succsim'^{\star}$. In both cases, $(x, y) \in \succsim'^{\star} \vee (y, x) \in \succsim'^{\star}$.

Let us prove that it is transitive. Let us consider $x, y, z \in \mathbb{D}$, such that $(x, y) \in \succsim'^{\star}$ and $(y, z) \in \succsim'^{\star}$. Therefore, $\exists i, k, m \in [\![1, d]\!]$ and $\exists j, l, n \in [\![1, e]\!]$ such that $x \in K_i, C_j$, $y \in K_k, C_l$, $z \in K_m, C_n$, $(i < k) \vee (i = k \wedge j \leq l)$ and $(k < m) \vee (k = m \wedge l \leq n)$. This is decomposed into four cases. Firstly, if $(i < k)$ and $(k < m)$, then $i < m$ and $(x, z) \in \succsim'^{\star}$. Secondly, if $(i < k)$ and $k = m$, then $i < m$ and $(x, z) \in \succsim'^{\star}$. Thirdly, if $i = k$ and $(k < m)$, then $i < m$ and $(x, z) \in \succsim'^{\star}$. Lastly, if $(i = k \wedge j \leq l)$ and $(k = m \wedge l \leq n)$, then $(i = m \wedge j \leq n)$ and $(x, z) \in \succsim'^{\star}$. It is always transitive. ☐

Proposition 19. $\succsim'^{\star} \in Ext(\succsim')$

Proof. Let us prove that $\succsim'^{\star}$ is an extension of $\succsim'$. Let us prove $I(\succsim') \subseteq I(\succsim'^{\star})$ and $P(\succsim') \subseteq P(\succsim'^{\star})$. Assume $(x, y) \in I(\succsim')$, then $\exists C_i \in \mathbb{C}$ such $x, y \in C_i$. By construction $\exists j \in [\![0, d]\!]$ such that $x, y \in K_j$. Knowing that $j \leq j$ and $i \leq i$, we get $x \succsim'^{\star} y$ and $y \succsim'^{\star} x$, therefore $I(\succsim') \subseteq I(\succsim'^{\star})$.

Assume $(x, y) \in P(\succsim')$, thus $\exists C_i, C_j \in \mathbb{C}$ such that $x \in C_i$, $y \in C_j$ and $C_i \succ_{\mathbb{C}} C_j$. Then, we can compare their longest path to a decision of a maximal class $\succ_{\mathbb{C}}$: $d_{\succ_{\mathbb{C}}}(x) + 1 \geq d_{\succ_{\mathbb{C}}}(y)$. Therefore $\exists a, b \in [\![0, d]\!]$ such that $x \in K_a, y \in K_b$ and $a < b$. By construction of $\succsim'^{\star}$, we get $(x, y) \in P(\succsim'^{\star})$. $\succsim'^{\star}$ is an extension of $\succsim'$. ☐

Proposition 20. *$\succsim'^{\star}$ satisfy W*

Proof. As $\succsim'^{\star}$ is a total preorder, it holds that $\mathbb{D}^2 = P(\succsim'^{\star}) \cup I(\succsim'^{\star}) \cup P(\succsim'^{\star})^{-1}$, as $W \subseteq \mathbb{D}^2$ it holds that $W \subseteq P(\succsim'^{\star}) \cup I(\succsim'^{\star}) \cup P(\succsim'^{\star})^{-1}$. Let us prove this proposition by proving that $W \cap P(\succsim'^{\star})^{-1} = \emptyset$. Ad absurdum, let us suppose that $\exists x, y \in \mathbb{D}$ such that $(y, x) \in W$ and $(x, y) \in P(\succsim'^{\star})$. As $x, y \in P(\succsim'^{\star})$, from $(x, y) \in \succsim'^{\star}$, we get that $\exists i, k \in [\![1, d]\!]$ and $j, l \in [\![1, e]\!]$ such that $x \in K_i, C_j$ and $y \in K_k, C_l$ and $(i < k) \vee (i = k \wedge j \leq l)$. From $(y, x) \notin \succsim'^{\star}$, we get $(k \leq i) \wedge (k \neq i \vee l > j)$. Combining this formula, we get $(k > i) \vee (k = i \wedge l > j)$.

Let us consider the case where $k > i$: it means that $d_{\succ_{\mathbb{C}}}(x) < d_{\succ_{\mathbb{C}}}(y)$, thus $\exists C_p \in max_{\succ_{\mathbb{C}}}$ such that there is a path in $\succ_{\mathbb{C}}$ from C_p to C_l of size k. Therefore, noting $y = b_k$, $\exists a_0, \dots, a_k, b_0, \dots, b_{k-1}$ and $\exists C_1, \dots, C_{k-1}$ with $a_0, b_0 \in C_p$, $\forall f \in [\![1, k-1]\!], a_f, b_f \in C_f$, $a_k \in C_l$ such that $\forall f \in [\![0, k-1]\!], a_f P(\succsim') b_{f+1}$. By construction of $\succsim'$, we get $\forall f \in [\![0, k-1]\!], a_f N'_{\succ} b_{f+1}$. Let us focus on the link between C_{k-1} and C_k: $a_{k-1} N'_{\succ} b_k$ means that $\exists g, h \in \mathbb{D}$, $a_{k-1} U^+ g U_{pr} h U^+ b_k$. As $b_k W x$ and by definition of U, $W \subseteq U$, it holds that $h U^+ x$. By definition of $N'_{\succ}$, it holds that $a_{k-1} N'_{\succ} x$ and thus $a_{k-1} P(\succsim') x$. By definition of $\succ_{\mathbb{C}}$, $C_{k-1} \succ_{\mathbb{C}} C_j$. Therefore there is a path in $\succ_{\mathbb{C}}$ from C_p to C_j of size k. As i is the size of the longest path from a maximal classes to C_j, we get $i \geq k$, which is absurd.

Let us consider the case where $(k = i \wedge l > j)$: by construction, we defined that $\forall k \in [\![1, d]\!], \forall a, b \in K_k$ such that $a \in C_i, b \in C_j$ and $i \neq j$, it holds that $(aWb \Rightarrow i < j)$. As yWx, all the conditions hold and thus $j < l$. It is absurd.

In both cases it is absurd, thus $W \subseteq P(\succsim'^{\star}) \cup I(\succsim'^{\star})$. ☐

To prove that $\succsim'^{\star}$ satisfy D, the following condition is necessary: $D \cap N'_{\sim} = \emptyset$. This is due to the fact that D can be considered as the negation of equivalence.

Proposition 21. *$\succsim'^{\star}$ satisfy D*

Proof. As $\succsim'^{\star}$ is a total preorder, it holds that $\mathbb{D}^2 = P(\succsim'^{\star}) \cup I(\succsim'^{\star}) \cup P(\succsim'^{\star})^{-1}$, as $D \subseteq \mathbb{D}^2$ it holds that $D \subseteq P(\succsim'^{\star}) \cup I(\succsim'^{\star}) \cup P(\succsim'^{\star})^{-1}$. By condition, it holds that $D \cap N_{\sim} = \emptyset$. Let us prove that $I(\succsim'^{\star}) \subseteq N'_{\sim}$. Let us suppose $\exists x, y \in \mathbb{D}$ such that $(x, y) \in I(\succsim'^{\star})$. It means that $(x, y) \in \succsim'^{\star}$ and $(y, x) \in \succsim'^{\star}$. From $(x, y) \in \succsim'^{\star}$, we get that $\exists i, k \in [\![1, d]\!]$ and $j, l \in [\![1, e]\!]$ such that $x \in K_i, C_j$ and $y \in K_k, C_l$ and $(i < k) \vee (i = k \wedge j \leq l)$. As $(y, x) \in \succsim'^{\star}$, it also holds that $(k < i) \vee (i = k \wedge l \leq j)$. Therefore not only $i = k$ but $j = l$. Thus $x, y \in C_j$. By definition of the equivalence classes, $(x, y) \in N'_{\sim}$. Thus, it holds that $D \cap I(\succsim'^{\star}) = \emptyset$. It proves that $D \subseteq P(\succsim'^{\star}) \cup P(\succsim'^{\star})^{-1}$. ☐

This concludes the proof. Three conditions have been assumed: $N'_{\succ}$ is asymmetric, $N'_{\sim} \cap N'_{\succ} = \emptyset$ and $D \cap N'_{\sim} = \emptyset$. Under this condition, this section builds $\succsim'^{\star} \in \bigcap_{i=1}^{n} TExt(R_i)$ such that W, D are satisfied by $\succsim'^{\star}$.

6 Conclusion

This paper proposes a definition of compatibility between preorders interpreted as preference relations on decisions. It establishes a necessary and sufficient condition of compatibility, which is a modified version of the consistency property proposed by Suzumura, using known theorems on the existence of an extension. Then, considering a finite set of decisions, as commonly occurs in the computational ethics settings, we propose an alternative proof of the sufficiency of the condition. This proof is constructive because it proposes four steps to construct a unique preference relation in which all preferences and equivalences defined in the preorders are preserved. Finally, we show that these steps can also be used to prove the compatibility of preorders with additional constraints.

Future works aim at applying the theoretical foundations proposed in this paper to computational ethics, as this enables the study of the compatibility between several ethical principles. In the case the principles are compatible, the extension proposed in Section 4 provides an effective representation of the ethical aspect of the decision problem. Ongoing works aim at determining the complexity of the construction of this extension. In case the principles are incompatible, future works aim at explaining the incompatibility sources so as to determine the extent to which they can be used jointly.

References

[1] W. Bossert. Incommensurability and consistency. *Philosophical Studies*, pages 1–17, 2023.

[2] R. Bradley. A note on incompleteness, transitivity and suzumura consistency. *Individual and Collective Choice and Social Welfare: Essays in Honor of Nick Baigent*, pages 31–47, 2015.

[3] P. Eades and L. Xuemin. How to draw a directed graph. In *1989 IEEE Workshop on Visual Languages*. IEEE Computer Society, 1989.

[4] P. C. Fishburn. *The Theory of Social Choice*. Princeton University Press, 1973.

[5] B. Hansson. Choice structures and preference relations. *Synthese*, 18 (4):443–458, 1968.

[6] P. Healy and N. S. Nikolov. Hierarchical drawing algorithms. In R. Tamassia, editor, *Handbook on Graph Drawing and Visualization*, pages 409–453. Chapman and Hall/CRC, 2013.

[7] N. Houy. A note on the Suzumura-consistency. *Mathematical Social Sciences*, 55(1):90–95, 2008. ISSN 0165-4896.

[8] J. L. Kelley. *General topology*. Courier Dover Publications, 2017.

[9] K. Mehlhorn. *Graph algorithms and NP-completeness*. Springer-Verlag, Berlin, Heidelberg, 1984.

[10] F. Rossi. Moral preferences. In *10th Workshop on Advances in Preference Handling (MPREF)*, 2016.

[11] K. Suzumura. Remarks on the theory of collective choice. *Economica*, pages 381–390, 1976.

[12] E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae*, 16(1):386–389, 1930.

[13] S. Tolmeijer, M. Kneer, C. Sarasua, M. Christen, and A. Bernstein. Implementations in machine ethics: A survey. *ACM Comput. Surv.*, 53(6), 12 2021.