Equitable Mechanism Design for Facility Location

Toby Walsh[*]

AI Institute, UNSW Sydney, Australia

Abstract. We consider strategy proof mechanisms for facility location which maximize equitability between agents. As is common in the literature, we measure equitability with the Gini index. We argue why the Gini index of agent utilities is a better measure of equitability than the Gini index of distances agents must travel to the nearest facility. We first prove a simple but fundamental impossibility result that no strategy proof mechanism can bound the approximation ratio of the optimal Gini index of utilities (or of distances) for one or more facilities. We propose instead computing approximation ratios of the complemented Gini index of utilities, and consider how well both deterministic and randomized mechanisms approximate this. For deterministic mechanisms and a single facility, we prove that the MEDIAN mechanism 2-approximates this objective, but that the MIDORNEAREST mechanism does even better, providing a $^6/_5$-approximation. This leaves a small gap to a lower bound on the approximation ratio for any deterministic and strategy proof mechanism of $^8/_7$. For randomized mechanisms and a single facility, we prove that the LRM mechanism (which is optimal with respect to approximating the maximum distance) is not optimal with respect to approximating the complemented Gini index of utilities. We also extend these approximability results to multiple facilities. For instance, we propose a new mechanism for locating two facilities with an approximation ratio of the optimal complemented Gini index of utilities that is better than the ENDPOINT mechanism, the only mechanism with a bounded approximation ratio of the minimum utility or maximum distance. Experiments demonstrate that these mechanisms perform well not just in the worst case but on average, often returning solutions within a few percent of optimal.

1 Introduction

Mechanism design is the problem of designing rules for a game to achieve a specific outcome, even though each participant may be self-interested. The aim is to design rules so that the participants are incentivized to behave as the designer intends. This typically includes achieving properties such as truthfulness, individual rationality, budget balance, and maximizing social welfare. Here we consider another desirable property that designers might look to achieve: equitability. How does a mechanism designer ensure that all participants are equally happy with the outcome? Surprisingly, equitable mechanism has received somewhat limited attention so far in the social choice literature.

Central to this question of equitable mechanism design is defining what it means for an outcome to be equitable. Consider the simple decision making problem of locating a facility along a line. This models a number of real world problems such as picking the room temperature for a classroom, or the deadline for a project. Agents have single peaked preferences, and their preference increases as the facility is moved nearer to their location. Our goal is to design mechanisms which locate the facility so that the distances which the different agents must travel are as similar as possible.

In general, of course, the distances agents travel often cannot be the same. Consider locating a facility on the interval $[0, 1]$, with three agents: one at 0, another at $^1/_2$ and the final at 1. The agent at $^1/_2$ inevitably has to be nearer the facility than at least one of the other two agents. There is no place on $[0, 1]$ that the facility can be located that ensures all three agents are the same distance from the facility. Where then do we locate the facility to ensure the most equitable outcome? In this case, locating the facility at $^1/_2$ might seem best. The agents at the two extremes both have to travel an equal distance, while the third agent is even better off. Any other solution is more inequitable as one of the agents at the endpoints must travels a greater distance, while the agent at the other endpoint travels less.

Our goal then is to design equitable mechanisms for facility location in which agents are incentivized to report sincerely. While our focus is on the facility location problem, there are some general conclusions that can be drawn from this study. First, designing mechanisms for equitability is somewhat different to designing mechanisms for other objectives such as social welfare. For instance, equitability considers all agents, while egalitarian welfare considers just the worst off agent. We will show that mechanisms with approximate well the egalitarian welfare may not return very equitable outcomes. Second, designing mechanisms for equitability is possible if we sacrifice a little optimality. Indeed, we identify mechanisms that are typically within a few percent of optimal, and come with worst case bounds not much greater than this. And third, equitability is not incompatible with the self-interest of participants. In particular, we identify strategy proof mechanisms that return outcomes with close to optimal equitability. We conjecture that equitable mechanism design may therefore offer promise in other domains such as fair division and ad auctions.

2 Facility location

The facility location problem is a classic problem in social choice (and mulitagent decision making) in which we need to decide where to locate a facility to serve a set of agents. We consider n agents located on the real line at x_1 to x_n . Without loss of generality, we suppose $x_1 \leq \ldots \leq x_n$. A deterministic mechanism f locates the facility at a location y. Formally, $f(x_1, \ldots, x_n) = y$. We let d_i be the distance of agent i to the facility: $d_i = |x_i - y|$. A randomized mechanism returns a probability distribution of facility locations. A mechanism is *strategy proof* iff no agent can misreport their position

[*] Email: tw@cse.unsw.edu.au. Supported by the Australian Research Council through an ARC Laureate Fellowship FL200100204.

and have the (expected) distance to the facility reduce. We limit our attention to *unanimous* mechanisms that when $x_1 = \ldots = x_n$, locate the facility at x_1. As in a number of previous studies, we assume that agents and facilities are on the finite interval $[0, 1]$, and the utility of agent i is $1 - d_i$. The interval could be $[a, b]$ supposing we normalise by $b - a$.

Supposing agents and facilities lie on a finite interval is interesting for both practical and theoretical reasons. In particular, agents and facilities are often limited to a finite interval and cannot be located outside those limits. For example, when setting a thermostat, we have a temperature range limited by the boiler. As a second example, when locating a water treatment plant on a river, the plant must be on the river itself. As a third example, when locating a shopping centre, the centre might have to be on the fixed (and finite) road network. There are thus many situations where agents are limited to a finite interval. Restricting agents to a finite interval also limits the extent to which agents can misreport their location to influence the outcome. A finite interval has been used in a number of recent works (e.g. [15, 6, 8, 2, 3]).

We consider various strategy proof mechanisms for facility location. The LEFTMOST mechanism locates the facility at x_1, the leftmost agent. The MEDIAN mechanism locates the facility at $x_{\lfloor n/2 \rfloor}$, the median agent. The MIDORNEAREST mechanism locates the facility at x_n when $x_n < 1/2$, at $1/2$ when $x_1 \leq 1/2 \leq x_n$, and at x_1 when $x_1 > 1/2$. The ENDPOINT mechanism locates one facility at x_1 and another at x_n. Agents are served by their nearest facility. These mechanisms are all deterministic. We also consider randomized mechanisms which return a lottery over solutions. For example, the LRM mechanism (Left Right or Midpoint) locates the facility at x_1 with probability $1/4$, at $(x_1 + x_n)/2$ with probability $1/2$, and at x_n with the remaining probability $1/4$. The mechanism is strategy proof. Agents cannot reduce their expected distance to the facility by misreporting their location.

We consider how well mechanisms approximate some objective O like the (soon to be defined) Gini index of distances. For a maximization objective, the approximation ratio is the maximum ratio of O_{opt}/O_{approx} where O_{opt} is the optimal objective value and O_{approx} is the approximately optimal objective value returned by the mechanism. For a minimization objective, the approximation ratio is the maximum ratio of O_{approx}/O_{opt}. For instance, a mechanism 2-approximates an objective iff the approximate solution returned by the mechanism is always within a factor of 2 of the optimal.

While our results are focused on the 1-d setting, they are interesting more broadly. The 1-d facility location problem models several real world problems such as locating ferry stops along a river or distribution centres along a highway. There are also non-geographical settings that are 1-d (e.g. setting a thermostat). In addition, we can solve more complex problems in higher dimensions by decomposing them into 1-d problems. Finally, the 1-d problem is the starting point to consider more complex metrics such as trees and networks.

3 Minimizing the Gini index

The Gini index is one of the most widely used measures of equitability in economics. It can be justified axiomatically in a number of ways. For instance, it is the unique index that satisfies scale invariance, symmetry, proportionality and convexity in similar rankings. Unsurprisingly it has been used in facility location problems. For example, Mulligan [17] argues that simple equitability measures like maximum distance ignore the distribution of distances and recommends instead measures like the Gini index. The Gini index of distances is defined by:

$$G_d \quad = \quad \frac{\sum_{i \leq n} \sum_{j \leq n} |d_i - d_j|}{2n \sum_{i \leq n} d_i}$$

This lies in $[0, 1]$, takes the value 0 for an equitable solution when $d_i = d_j$ for all i and j, and increases in value as distances become more unequal. If all agents are at the same location, then any facility location is an equitable solution since all agents travel the same distance. Therefore equitability alone is not sufficient to guarantee solutions are desirable. We might also demand additional properties like unanimity.

Unfortunately, strategy proof mechanisms cannot approximate well the optimal Gini index of distances. We begin with a simple but important impossibility result.

Theorem 1. *No strategy proof mechanism for locating one or more facilities on the real line or on the interval* $[0, 1]$ *has a bounded approximation ratio for the Gini index of distances with any number of agents.*

Proof. Suppose there exists a strategy proof mechanism with a bounded approximation ratio for locating a facility. Consider one agent is at 0 and the second at $1/2$. The Gini index takes the value zero iff the facility is located at the mean location, $1/4$. To satisfy the approximation ratio, the facility must be located at this position. Suppose the second agent reports location 1. To satisfy the approximation ratio, the facility must now be located at the new mean location, $1/2$. Hence, if agents are at 0 and $1/2$ then the agent at $1/2$ has an incentive to mis-report their location as 1. This contradicts our assumption that there exists a strategy proof mechanism with a bounded approximation ratio.

Suppose there exists a strategy proof mechanism with a bounded approximation ratio for locating n facilities ($n \geq 2$). Consider agents at 0, $1/16n$ and $3/2n$, $5/2n$, $\ldots$, $1 - 1/2n$. The Gini index takes its zero value when facilities are located at $1/32n$, and $3/2n \pm 1/32n$, $5/2n \pm 1/32n, \ldots, 1 - 1/2n \pm 1/32n$. To satisfy the approximation ratio, the mechanism must locate the n facilities at these n locations. Suppose the second agent reports location $1/8n$. To satisfy the approximation ratio, the leftmost facility must now be located at $1/16n$. Hence, if agents are at 0, $1/16n$ and $3/2n$, $5/2n$, $\ldots$, $1 - 1/2n$ then the agent at $1/16n$ has an incentive to mis-report their location as $1/8n$. This contradicts our assumption that there exists a strategy proof mechanism with a bounded approximation ratio. Note that randomization does not help escape this impossibility. The proof works whether mechanisms are deterministic or randomized. □

4 Switching to utilities

But does minimizing the Gini index of distances even guarantee an equitable outcome? Minimizing the Gini index of incomes rightly favours incomes being large over incomes being small. But minimizing the Gini index of distances perversely favours distances being large over distances being small. As an extreme example, in Figure 1 we have one agent at 0 and $n - 1$ agents at 1 for $n > 2$. If we minimize the Gini index of distances then locating the facility at 0 is preferred, giving the minimum possible Gini index of $1/n$, compared to locating the facility at 1, giving the maximum possible Gini index of $(n - 1)/n$. Thus, on this problem, minimizing the Gini index of distances prefers the solution in which most agents travel the maximum possible distance and one agent has no distance to travel, over the solution where one agent has to travel the maximum possible distance but most agents have not to travel at all.

A better and more equitable solution in facility location is to consider utilities rather than distances and an equitability measure like the Gini index of utilities.

$$G_u \quad = \quad \frac{\sum_{i \leq n} \sum_{j \leq n} |u_i - u_j|}{2n \sum_{i \leq n} u_i}$$

Minimizing the Gini index of utilities prefers most agents having the maximum possible utility over most agents having the minimum possible utility. In the example in Figure 1, minimizing the Gini index of utilities would favour locating the facility at 1 along with most agents, rather than at the single agent located at 0.

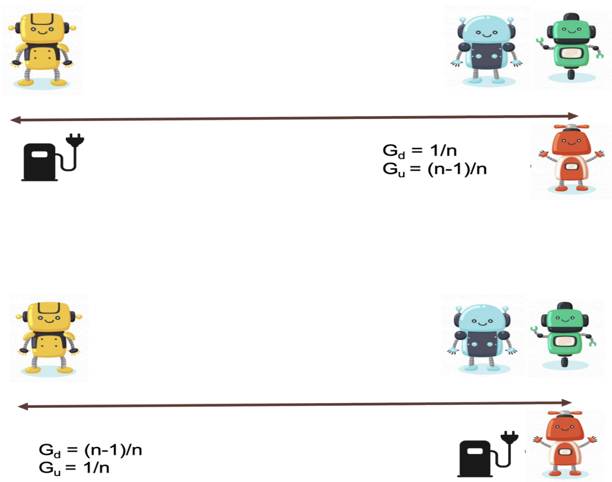

Figure 1: Example of facility location problem with n agents illustrating why minimizing the Gini index of utilities (G_u) rather than the Gini index of distances (G_u) gives an equitable outcome in which agents travel the least.

Unfortunately, switching to the Gini index of utilities rather than the Gini index of distances does not also fix the problem that approximation ratios can be unbounded.

Theorem 2. *No strategy proof mechanism for locating one or more facilities on* $[0, 1]$ *has a bounded approximation ratio for the Gini index of utilities with any number of agents.*

Proof. We can simply repeat the previous proof. □

The problem with the approximation ratio of the Gini index, whether it be the Gini index of utilities or of distances, is that the approximation ratio of the Gini index concentrates attention on equitable problems where the index is zero (and all distances/utilities are equal). A natural way around this problem is to consider the complemented Gini index (that is, $1 - G$). This is again in $[0, 1]$ but now is 1 when utilities (or distances) are equal. Our goal now is to *maximize* the complemented Gini index rather than *minimize* the original Gini index. Considering the approximation ratio of the complemented Gini index switches attention away from equitable problems in which utilities (or distances) are balanced to inequitable problems in which utilities (or distances) are necessarily imbalanced (such as the example in the introduction with agents at 0, $1/2$ and 1). You might be concerned that by shifting to the complement of the Gini index, we are just replacing one problem (approximating equitable problems with Gini indices close to zero) with another (approximating inequitable problems with Gini indices close to 1). This is not the case. While Gini indices can indeed be close to zero (and hard to approximate within a constant factor), it is not hard to show that, with a facility location problem, the *optimal* Gini index of utilities is never that close to 1 (and, as we shall show, can be approximated well).

5 Deterministic mechanisms

Now that we have identified an appropriate objective (namely, the complemented Gini index of utilities), we prove that there exist strategy proof mechanisms which approximate this objective well. We start with one of the simplest possible mechanisms. The LEFTMOST mechanism is strategy proof and 2-approximates the maximum distance [18]. This is optimal as no deterministic and strategy proof mechanism can do better [18]. The LEFTMOST mechanism is worse at approximating the complemented Gini index. By Theorems 1 and 2, there is also no bound on how poorly it approximates the Gini index of distances or utilities as it is strategy proof. While it has a bounded approximation ratio for the complemented Gini index of distances, the bound is not very good (and increases with the number of agents).

Theorem 3. *For a facility location problem with n agents, the* LEFTMOST *mechanism n-approximates the complemented Gini index of utilities.*

Proof. If all agents are at the same location, then any mechanism is optimal wrt both the Gini index and the (complemented) Gini index. Therefore we suppose agents are at two or more locations. The smallest possible complemented Gini index (which the LEFTMOST mechanism achieves) is when one agent is at 0 and the remaining $n - 1$ agents are at 1. The complemented Gini index in this case is just $1/n$. Note that it is impossible for the Gini index to be larger (and thus for the complemented Gini index to be smaller). The optimal complemented Gini index that can be achieved in this case is, on the other hand, maximal. In particular, if we locate the facility at $1/2$, the complemented Gini index is 1. The LEFTMOST mechanism therefore n-approximates the optimal complemented Gini index. □

The MEDIAN mechanism does significantly better than the LEFTMOST mechanism. This is unsurprising as the MEDIAN mechanism is more balanced than the LEFTMOST mechanism which locates the facility at an extreme location. When considering the approximability of egalitarian welfare, we cannot distinguish between the LEFTMOST and MEDIAN mechanisms. Both 2-approximate the maximum distance. Here we see that equitability distinguishes them apart.

Theorem 4. *The* MEDIAN *mechanism* 2*-approximates the complemented Gini index of utilities.*

Proof. We prove first that the MEDIAN mechanism always returns a facility location that gives a complemented Gini index of utilities of $1/2$ or greater. As the complemented Gini index is 1 or smaller, the MEDIAN mechanism therefore cannot be worse than a 2-approximation. Indeed, if we have one agent at 0 and another at 1, we note that it is at best a 2-approximation.

Suppose agents are at x_1 to x_n with $x_1 \leq \ldots \leq x_n$. Without loss of generality, we suppose $x_1 = 0$. There are two cases. In the first case, $n = 2k$ is even. Note that the median agent is at x_k, which is the location of the facility. WLOG we suppose $x_k \leq 1/2$ otherwise we reflect the position of any agent x onto $1 - x$ (and again if necessary shift all agents to left so leftmost agent is at 0). We first prove that $\sum_i u_i$ takes a minimum value of k. In fact, the minimum value of this sum is when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$. Suppose there is a smaller value for the sum of utilities for some other values x'_1 to x'_n. If we map x'_i onto 0 for $i \leq k$ then each u_i for $i \leq k$ increases less than each u_i for $i > k$ decreases. That is, the sum of utilities would decrease which is a contradiction. Hence, $x'_1 = \ldots = x'_k = 0$. Similarly, suppose $x'_{k+1} < 1$. Then mapping x_i onto 1 for $i > k$ would decrease the sum of utilities which is again a contradiction. Hence, the smallest sum of utilities occurs when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$ and this sum is k.

We next prove that $\sum_i \sum_j |u_i - u_j|$ takes a maximum value of $2k^2$. Again, the maximum value of this double sum is when $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$. We consider different terms in the double sum. If we consider the pair of terms $|u_i - u_j| + |u_i - u_{n-j+1}|$ for $i < j \leq k$ then as x_i and x_j are at or to the left of x_k, and at or the right of x_k, it follows that the sum of these two differences equals or is less than 1. A similar argument applies to the sum of terms $|u_{n-i+1} - u_{n-j+1}| + |u_{n-i+1} - u_j|$. Hence the $4k^2$ terms have a maximum sum of $2k^2$. This largest double sum occurs when again $x_1 = \ldots = x_k = 0$ and $x_{k+1} = \ldots = x_n = 1$.

The complemented Gini index is $1 - \frac{\sum_i \sum_j |u_i - u_j|}{2n \sum_i u_i}$. The minimum value this takes is lower bounded by the maximum value of $\sum_i \sum_j |u_i - u_j|$ divided by the minimum value of $\sum_i u_i$. That is a lower bound of $1 - \frac{2k^2}{4k \cdot k}$ or $1/2$.

In the second case $n = 2k+1$ is odd. WLOG we suppose $x_{k+1} \leq 1/2$. This is the median agent and therefore location of the facility. By a similar argument, $\sum_i u_i$ takes a minimum value of $k+1$, and $\sum_i \sum_j |u_i - u_j|$ takes a maximum value of $2k(k+1)$ when $x_1 = \ldots = x_{k+1} = 0$ and $x_{k+2} = \ldots = x_n = 1$ The complemented Gini index takes its minimum value of $1 - \frac{2k(k+1)}{2(2k+1)(k+1)}$ or $1 - \frac{k}{2k+1}$ which tends to $1/2$ from above as k goes to infinity. □

Can we do even better than this? Consider the MIDORNEAREST mechanism. This is strategy proof and 2-approximates the optimal maximum distance (and no strategy proof and deterministic mechanism can do better). It also $3/2$-approximates the optimal minimum utility (and no strategy proof and deterministic mechanism can do better) [1]. We now show that the MIDORNEAREST mechanism provides a close to optimal equitable solution.

Theorem 5. *The* MIDORNEAREST *mechanism $6/5$-approximates the optimal complemented Gini index of utilities.*

Proof. Observe that the MIDORNEAREST mechanism guarantees that $u_i \geq 1/2$ for any i. The most inequitable outcome for this mechanism then is when $u_i = 1/2$ for $i < n$ and $u_n = 1$. This occurs, for example, when $x_i = 0$ for $i < n$ and $x_n = 1/2$, and the MIDORNEAREST mechanism locates the facility at $1/2$. This gives a Gini index of utilities of $\frac{(n-1)}{n(n+1)}$. This is maximized for $n = 2$ and $n = 3$ when the Gini index is $1/6$. The complement of the Gini index is thus $5/6$ or greater. Coincidently, when $x_i = 0$ for $i < n$ and $x_n = 1/2$, there is an optimal and perfectly equitable solution which locates the facility at $1/4$, giving a complemented Gini index of utilities of 1. Hence, the most inequitable outcome for the MIDORNEAREST mechanism with a Gini index of $1/6$ occurs when there is an optimal and perfectly equitable solution. The approximation ratio of the MIDORNEAREST mechansm is thus $6/5$. □

It is easy to see that any strategy proof mechanism must approximate the optimal complemented Gini index of utilities. For instance, with two agents, a mechanism that returns the optimal complemented Gini index would need to track the midpoint between the two agents which is not strategy proof. In fact, we can show that no strategy proof mechanism can do better than a constant factor approximation.

Theorem 6. *No deterministic and strategy proof mechanism for locating a facility can do better than $8/7$-approximate the optimal complemented Gini index of utilities.*

Proof. Suppose there exists a strategy proof mechanism that provides better than a $8/7$-approximation. Consider two agents, one at $1/3$ and another at $2/3$. There are two cases. In the first case, the mechanism locates the facility in $[0, 1/2]$. The second case, when the mechanism locates the facility in $(1/2, 1]$ is dual and we need not consider further. Suppose the agent at $2/3$ mis-reports their location as 1. An optimal mechanism now puts the facility at $2/3$ giving a complemented Gini index of 1. To achieve the approximation ratio, the complemented Gini index must again be greater than $7/8$. This puts the facility in the interval $(1/2, 5/6)$. Note that this is now closer to the actual location of the agent at $2/3$ than the previous location of the facility in $[0, 1/2]$. Hence, this agent has an incentive to misreport. This contradicts the assumption that the mechanism is strategy proof. □

There remains a small gap beween the approximation ratio of $6/5$ achieved by the MIDORNEAREST mechanism and this lower bound of $8/7$. It is an interesting open problem to close this gap.

6 Randomized mechanisms

Randomization is often a simple and attractive mechanism to achieve better performance. For example, the randomized LRM mechanism $3/2$-approximates the maximum distance any agent must travel in expectation. This is a better approximation ratio than deterministic mechanisms can achieve as no such strategy proof mechanism can do better than 2-approximate the maximum distance. Indeed, the LRM mechanism is optimal wrt optimizing the maximum distance as no randomized and strategy proof mechanism can do better. The LRM mechanism does a less good job at approximating the complemented Gini index of utilities.

Theorem 7. *The* LRM *mechanism $40/27$-approximates the optimal complemented Gini index of utilities in expectation (≈ 1.48).*

Proof. For a single agent, the LRM mechanism is optimal and locates the facility at the agent. The complemented Gini index of utilities is then 1 which is optimal.

For two agents, we suppose one agent is at 0 and the other at a with $0 \leq a \leq 1$. With probability $1/2$, the facility is located at $a/2$ which gives an optimal complemented Gini index of utilities of 1. With the remaining probability, the facility is located at 0 or a which gives a sub-optimal complemented Gini index of utilities of $1 - \frac{a}{2(2-a)}$. This is minimized for $a = 1$ when the complemented Gini index of utilities is $1/2$. The LRM mechanism thus has an expected complemented Gini index of utilities that is $3/4$, compared to an optimal of 1. Thus, it $4/3$-approximates the optimal.

For three agents, we suppose one agent is at 0, another at a and the third at b with $0 \leq a \leq b \leq 1$. Without loss of generality, we suppose $2a \leq b$ (otherwise we reflect problem). The Gini index of utilities is minimized when the facility is at $b/2$ and the complemented Gini index of utilities is $1 - \frac{4a}{3(6-3b+2a)}$. With probability $1/2$, the LRM mechanism locates the facility at $b/2$ which gives an optimal complemented Gini index of utilities of $1 - \frac{4a}{3(6-3b+2a)}$. With probability $1/4$, the facility is located at 0 which gives a sub-optimal complemented Gini index of utilities of $1 - \frac{2b}{3(3-a-b)}$. With the remaining probability $1/4$, the facility is located at b which gives a sub-optimal complemented Gini index of utilities of $1 - \frac{2b}{3(3+a-2b)}$. The expected complemented Gini index of utilities is thus $1 - \frac{2a}{3(6-3b+2a)} - \frac{b}{6(3-a-b)} - \frac{b}{6(3+a-2b)}$. The ratio of this with the optimal is maximized by $a = 1/2$ and $b = 1$ when the expected value is $25/36$ compared to an optimal of $5/6$. Hence the LRM mechanism $6/5$-approximates the optimal.

For four or more agents, we suppose without loss of generality that the leftmost agent is at 0 (i.e. $x_1 = 0$). Suppose the rightmost agent $x_n < 1$. Then if we map the agent at x_i to $\frac{x_i}{x_n}$, we will stretch the distribution of agents out, increasing any inequitability. Hence, the worst inequitability occurs when $x_n = 1$. We consider two cases. In the first, with probability $1/2$, the facility is located at $1/2$. Agents then must have utility $1/2$ or greater. The most inequitable case is when one agent has utility 1, and all other agents have utility $1/2$. This corresponds to one agent at $1/2$ and all other agents at 0 or 1, giving a Gini index of utilities of $\frac{(n-1)}{n(n+1)}$. Hence, the complemented Gini index of utilities is $1 - \frac{(n-1)}{n(n+1)}$ or greater. This is minimized for $n = 4$ when it gives a complemented Gini index of utilities of $17/20$. In the second case, with probability $1/2$, the facility is located at one of the end points. Let x_i be the location of agent i. The second case contributes $1/2 - 1/4(\frac{\sum_{i,j} |x_i - x_j|}{2n \sum_i x_i} + \frac{\sum_{i,j} |(1-x_i)-(1-x_j)|}{2n \sum_i (1-x_i)})$ to the expected complemented Gini index of utilities. This simplifies to $1/2 - \sum_{i,j} |x_i - x_j|/8n(\frac{1}{\sum_i x_i} + \frac{1}{\sum_i (1-x_i)})$. This is minimized for $x_1 = 0$, and $x_i = 1$ for $i > 1$ (or dually for $x_n = 1$ and $x_i = 0$ for $i < n$) and $n = 4$ when it takes the value $1/4$. Thus the expected complemented Gini index of utilities is at least $17/40 + 1/4$ or $27/40$. Thus the LRM mechanism is at least an $40/27$-approximation of the optimal complemented Gini index of utilities. □

While an approximation ratio of $40/27$ might seem reasonable, it is perhaps disappointing given that the deterministic MIDORNEAREST mechanism does better, achieving an approximation ratio of the optimal complemented Gini index of utilities of just $6/5$.

7 Two facility location

We next consider how these results extend to multiple facilities. With two facilities, the ENDPOINT mechanism is the only strategy proof and deterministic mechanism with a bounded approximation ratio of the optimal maximum distance or of the optimal minimum utility. More precisely, it 2-approximates the optimal maximum distance, and $3/2$-approximates the optimal minimum utility. As the ENDPOINT mechanism is strategy proof is does not bound the approximation ratio of the Gini index of distances or of utilities. With respect to the complemented Gini index of utilities, it offers a good approximation ratio of the optimal. However, somewhat surprisingly, it does not offer the best possible ratio amongst strategy proof and deterministic mechanisms.

Theorem 8. *The* ENDPOINT *mechanism $35/29$-approximates the optimal complemented Gini index of utilities (≈ 1.21).*

Proof. With two or fewer agents, the ENDPOINT mechanism returns an optimal solution in which both the utilities and the complemented Gini index of utilities are maximal. Therefore we consider three or more agents. With the ENDPOINT mechanism, the leftmost and rightmost agents must have utility 1, while the other agents have utility between $1/2$ and 1. Suppose there are n agents ($n \geq 3$) with utilities in $[1/2, 1]$, and two of the agents have utility 1. Then the minimum complemented Gini index of utilities (or equivalently the maximum Gini index of utilities) is when $n - 2$ agents have utility $1/2$, and two have utility 1. This occurs when one agent is at 0, another is at 1, and the final $n - 2$ are at $1/2$, with the facilities located at the two endpoints. In this case, the Gini index of utilities is $\frac{2(n-2)}{n(n+2)}$. This is maximized for $n = 5$, when it is $6/35$. The corresponding complemented Gini index of utilities is $29/35$. Coincidently, when one agent is at 0, another is at 1, and the final $n - 2$ are at $1/2$ there is an optimal, perfectly equitable outcome when we locate facilities at $1/4$ and $3/4$. The optimal complemented Gini index of utilities in this case is 1. This gives an approximation ratio of the complemented Gini index of utilities of $35/29$. □

We now define a new mechanism, the truncated ENDPOINT mechanism which performs better. For two or fewer agents, this simply applies the ENDPOINT mechanism. For three or more agents, if x_1 is the leftmost agent, and x_n is the rightmost then it locates the left facility at $\max(x_1, 1/4)$ and the right facility at $\min(3/4, x_n)$. The truncated ENDPOINT mechanism trivially retains the strategy proofness of the original. It also offers a better approximation ratio than the original ENDPOINT mechanism.

Theorem 9. *The truncated* ENDPOINT *mechanism $15/14$-approximates the optimal complemented Gini index of utilities (≈ 1.07).*

Proof. With two or fewer agents, the truncated ENDPOINT mechanism returns an optimal and perfectly equitable solution in which both the utilities and the complemented Gini index of utilities are maximal. Therefore we consider three or more agents. With the truncated ENDPOINT mechanism, all agents must have utility between $3/4$ and 1. Suppose there are n agents ($n \geq 3$) with utilities in $[3/4, 1]$. Then the minimum complemented Gini index of utilities (or equivalently the maximum Gini index of utilities) is when $n - 1$ agents have utility $3/4$ and the other agent has utility 1. This occurs when one agent is at 0, another is at $3/4$ and $n - 2$ are at $1/2$, and when the truncated ENDPOINT mechanism places facilities at $1/4$ and $3/4$. In this case, the Gini index of utilities is $\frac{(n-1)}{n(3n+1)}$. This is maximized for $n = 3$, when it is $1/15$. The corresponding complemented Gini index of utilities is $14/15$. Coincidently, when the agents are at 0, $1/2$ and $3/4$, there is an optimal, and perfectly equitable solution in which facilities are at $1/4$ and 1, and the complemented Gini index of utilities is 1. This gives an approximation ratio of the complemented Gini index of utilities for the truncated ENDPOINT mechanism of $15/14$. □

We now prove no strategy proof and deterministic mechanism for two facilities can do better than $30/29$-approximate the optimal complemented Gini index (≈ 1.03). This leaves a small gap with the approximation ratio of $15/14$ (≈ 1.07) achieved by the truncated ENDPOINT mechanism. It is an interesting open question to close this gap.

Theorem 10. *No strategy proof and deterministic mechanism for two facilities can do better than $30/29$-approximate the optimal complemented Gini index of utilities (≈ 1.03).*

Proof. We suppose a strategy proof and deterministic mechanism exists with an approximation ratio smaller that $30/29$. Consider three agents, one at 0, another at $1/2$ and the final agent at $3/4$. The optimal location of facilities that maximizes the complemented Gini index of utilities (or equivalently minimizes the Gini index of utilities) has one facility at $1/8$, and the other at $5/8$ giving a complemented Gini index of 1. The most right that the leftmost facility can be and the approximation ratio of the complemented Gini index be smaller than $30/29$ is to the left of $9/13$. If the rightmost facility is at $\frac{9}{13}$ then the minimal Gini index of utilities is when the leftmost facility is at $1/13$ and the Gini index is $1/30$. The complemented Gini index is then $29/30$, which corresponds to an approximation ratio of the optimal complemented Gini index of $30/29$. The agent at $3/4$ therefore travels a distance greater than $3/4 - 9/13$ (which is $3/52$).

Now suppose the agent at $3/4$ mis-reports their location as 1. The optimal location of facilities that maximizes the complemented Gini index of utilities (or equivalently minimizes the Gini index of utilities) for the reported locations of the agents has one facility at $1/4$, and the other at $3/4$ giving a complemented Gini index of 1. The most left that the leftmost facility can be and the approximation ratio of the complemented Gini index be smaller than $30/29$ is to the right of $9/13$. If the rightmost facility is at $\frac{9}{13}$ then the minimal Gini index of utilities for the reported locations is when the leftmost facility is at $5/26$ and the Gini index is $1/30$. The complemented Gini index is then $29/30$, which corresponds to an approximation ratio of the optimal complemented Gini index of $30/29$. Note also that the rightmost facility cannot be to the right of $3/4 + 3/52$ as this gives an approximation ratio of the complemented Gini index greater than $30/29$. The agent at $3/4$ therefore travels a distance less than $3/4 - 9/13$ (which is $3/52$). Thus, by mis-reporting their location, the agent at $3/4$ reduces their distance from the facility from more than $3/52$ to less than $3/52$. □

8 Experiments

These theoretical results on approximation ratios are all worst case. They provide bounds on how poorly a mechanism may perform. They do not inform us how well they tend to perform. We therefore ran some experiments to explore how well these mechanism generate equitable solutions. In each experiment, we generated 1024 instances at each problem size from 2^1 to 2^6 agents. Agents were located according to one of three different models: (1) uniformly on $[0, 1]$; (2) following a Bates distribution with parameter $k = 1$ to 10; or (3) according to a bimodal Kumaraswamy distribution.

The Bates distribution is a probability distribution of the mean of k statistically independent uniform random variables on the unit interval. For $k = 1$, it is simply the uniform distribution. For $k = 2$, it is the triangular distribution. For k large, it approaches a normal Gaussian distribution. The Kumaraswamy distribution has two shape parameters, a and b and is bimodal when $a = b = 1/2$. The Kumaraswamy distribution is similar to the Beta distribution, but is much simpler to use in simulation studies since its cumulative distribution function (CDF) has a simple closed form:

$$prob(Z \leq x) \quad = \quad 1 - (1 - x^a)^b = 1 - \sqrt{(1 - \sqrt{x})}$$

This choice of parameters ($a = b = 1/2$) tends to concentrate agents around either 0 or 1. See Figure 2 for details.

In Figure 2, we plot the approximation ratio observed for the MEDIAN, LEFTMOST and MIDORNEAREST mechanisms on problems where agents are located uniformly on $[0, 1]$. We observe that the MIDORNEAREST mechanism performs best, especially on smaller problems where the MEDIAN mechanism can perform less well. The approximation ratios for all of the mechanisms are significantly less than the worst case bounds. For instance, the MEDIAN

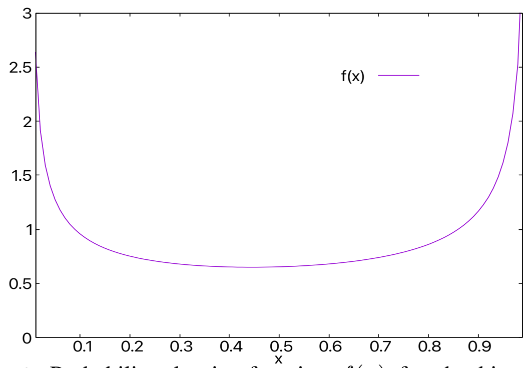

Figure 2: Probability density function $f(x)$ for the bimodal Kumaraswamy distribution used in the experiments.

mechanism, especially on larger problems, is much better than the 2-approximation realized in the worst case. In fact, solutions tend to be within a few percent of the optimal complemented Gini index. We observe similar results with problems coming from the Bates or bimodal Kumaraswamy distributions.

9 Related Work

Beginning with Procaccia and Tennholtz [20], most studies of strategy proof mechanisms for facility location have focused on approximating the total and maximum distance (e.g. [7, 9, 10, 11, 12, 22, 23]). Indeed, in a recent survey of mechanism design for facility location, strategy proof mechanisms which approximate well the total or maximum distance that agents travel has been called the "classic setting" [5]. One of the simplest measures of equitability in facility location is the maximum distance agents travel. Marsch and Schilling [14] claim that this "is the earliest and most frequently used measure that has an equity component" in facility location problems, that "it has long been used as a more equitable alternative to the p-median problem which minimizes [total] travel distance", and that it "quantifies the popular Rawlsian criteria of equity which seeks to improve as much as possible those who are worst-off".

Related to maximum distance is minimum happiness. The "happiness" of agent i is $h_i = 1 - d_i/d^i_{max}$ where d^i_{max} is the maximum possible distance agent i can travel [15, 16]. For instance, with agents and facilities constrained to the interval $[0, 1]$, $d^i_{max} = \max(x_i, 1 - x_i)$. This normalization changes the approximation ratios achievable compared to approximating just the maximum distance. For example, the MEDIAN mechanism, which 2-approximates the maximum distance, does not bound the minimum happiness of any agent.

Another common measure of equitability is variance. For example, Maimon [13] develops an algorithm to locate a facility on a tree network minimizing the variance in distances agents travel. Procaccia *et al.* [19] explore a different use of variance, exploring the trade-off in randomized mechanisms between variance in the distribution of the location a facility and the approximation ratio of the optimal total or maximum distance agents travel. Other simple measures of equitability are the range and absolute deviation in distances agents travel [14]. Berman and Kaplan [4], for example, argue that the latter is "a natural measure of the equity" of facility location problems and provide an efficient algorithm to compute the location of a facility on a general network to minimize this measure.

There are other indices of inequality besides the Gini index. For example, a common measure of income inequality is the Hoover index (also known as the Robin Hood or Schutz index), and this

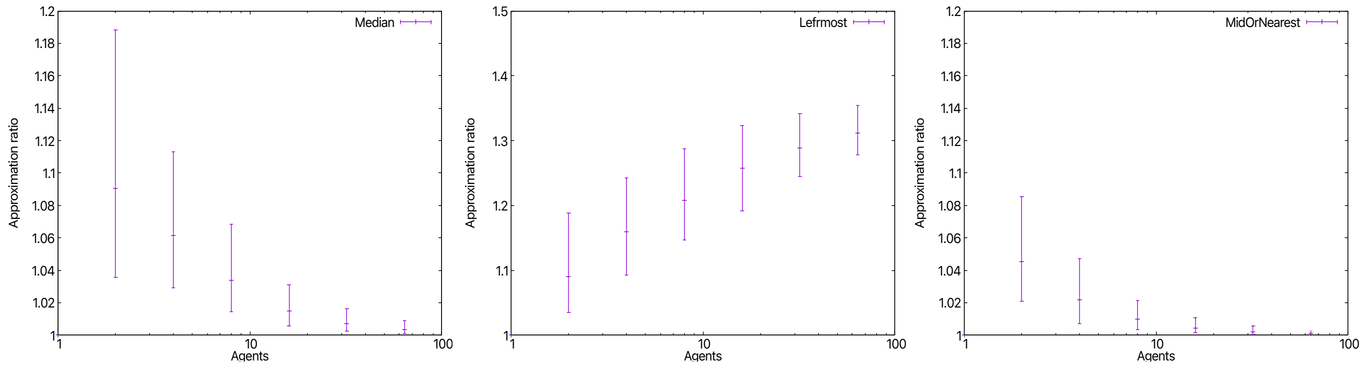

Figure 3: Median approximation ratio of the complemented Gini index of utilities for the deterministic MEDIAN, LEFTMOST and MIDORNEAREST mechanisms on uniform problem instances. Error bars give upper and lower quartile performance.

has been applied to facility location [17]. As a second example, the Atkinson index has been used in social choice settings such as resource allocation [21]. Like the Gini index, no strategy proof mechanism can approximate either index to within a constant factor.

10 Conclusions

We have proposed approximate mechanism design for equitability. For the facility location problem, we argued why the Gini index of agent utilities is a better measure of equitability than the Gini index of distances that agents travel. We first proved an impossibility result that strategy proof mechanism for one or more facilities cannot bound the approximation ratio of the optimal Gini index. We instead turn the problem on its head by considering approximation ratios of the complemented Gini index of utilities. For both deterministic and randomized mechanisms for a single facility, we identified mechanisms that bound the approximation ratio of this objective. In the case of randomized mechanisms, we construct a new mechanism with an optimal ratio. We then extended results to multiple facilities. For instance, we proposed a new strategy proof mechanism with a better approximation ratio for two facilities than the ENDPOINT mechanism, the only strategy proof mechanism with a bounded approximation ratio of the optimal minimum utility or maximum distance. Experiments showed that these mechanisms perform well, returning solutions close to optimal.

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