Method of Equal Shares with Dynamic Affordability

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Abstract. We introduce a new Participatory Budgeting rule similar to the method of equal shares (MES) that we call dynamic MES. Our results show that dynamic MES preserves the desirable theoretical guarantees of MES, including proportionality and efficiency properties, while showing better performance on real-world datasets from pabulib, especially when considering the rules without completion. However, the enhanced performance comes at the cost of significantly increased computational complexity, which currently limits its practical adoption compared to the original MES algorithm. Additionally, implementation challenges remain unresolved, requiring further development before dynamic MES can serve as a viable alternative in large scale participatory Budgeting instances.

1 Introduction

Participatory Budgeting (PB) is a democratic procedure that has gained a lot of interest and reached many cities in the past decades, we refer the reader to Dias et al. [4], Wampler et al. [21] for an exhaustive look at the development of PB in the world. The computational social choice community has shown a lot of interest for this problem, as documented and reviewed in Rey and Maly [15], Aziz and Shah [1]. In this article we explore approval based participatory budgeting, as it is the most commonly used and studied type of ballot for participatory budgeting.

The method of equal shares (MES) introduced by Peters et al. [13] is considered as one of the most if not the most interesting rule in Participatory Budgeting. The principle of this rule is to divide the total budget into equal shares given to every voters, then MES will allocate their individual budgets in order to pay for their approved projects. It gives strong axiomatic guarantees for a proportional outcome (EJR) and behaves well on real world examples provided in pabulib (a library of real world PB instances provided by Stolicki et al. [18]) as shown by Faliszewski et al. [7], Utke et al. [19]. Several adaptations and completions of MES have been developed in the past year with different objectives. Maly et al. [11] proposed a fair-share version of the rule that aimed at satisfying local fair share, a fairness axiom they introduced which MES fails to guarantee. They also measured experimentally how their new rule behaves compared to MES. Surprisingly, the "basic" version of MES gave slightly better experimental results than theirs on the metrics they defined in order to measure the distance to fair share.

One of the drawbacks of MES is that it often outputs a budget allocation that is not exhaustive (it is still possible to fund some unselected projects). In order to solve this problem, Peters et al. [13] proposed several completion methods. One can use another PB rule for with the remaining budget and unselected projects (usually Phragmén rule due to Phragmén [14], Brill et al. [3]). They recommend not doing so, for real world applications arguing that it is hard to explain to people why the second rule is not used from the beginning. Another way of making MES exhaustive is to raise the share of voters and re run MES until it is exhaustive with regard to the original budget limit. This is the most used MES completion rule in real world applications but has several drawbacks: some projects that are selected in the original MES run might not be selected in the end, it is then hard to explain why this is. Moreover, this method computes the MES algorithm from scratch several times, and it is thus not computationally efficient. Kraiczy and Elkind [8] defined a version of the rule that gives a more computationally efficient completion method for MES. They make it more adaptive so that every time the share is increased we can run MES from the previous allocation. In the same spirit, Kraiczy et al. [9] proposes another version of MES that had been implicitly proposed by Peters et al. [13], Kraiczy and Elkind [8]: Exact Equal Shares (EES). Its big advantage compared to MES is that its completion is easier to compute, but its efficiency and fairness has not been studied on experimental data.

Another drawback of the method of equal shares is the Helenka paradox introduced by Papasotiropoulos et al. [12]. In some critical real world examples, it can select a strongly sub optimal outcome. In order to avoid this type of paradox, they defined a new version of MES with bounded overspending where voters may spend more than their share of the budget. Their rule unlike the original MES fails EJR, but guarantees EJR up to $\frac{n-|S|}{2|S|}$, where $|S|$ is the size of a cohesive[2] group of voters. This rule has two big advantages compared to MES: it always outputs an exhaustive budget allocation, no longer needs a completion rule and avoids the Helenka paradox. Papasotiropoulos et al. [12] also show that although it does not guarantee EJR, MES with bounded overspending often satisfies it in real world examples (in approximately 95% of the cases).

Building on the work of Sánchez-Fernández et al. [17] that defined a dynamic version of Phragmén [14] sequential rule and of Maly et al. [11], we introduce a new version of MES: MES with dynamic affordability (or dynamic MES) with the aim of getting closer to fair share than the original MES rule. We study this rule both axiomatically and experimentally in order to present its advantages and limitations.

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[1] Equal contribution.

[2] A cohesive group is a group of voters with close preferences, to be defined in the preliminaries.

2 Preliminaries

We use the notations and definitions from Rey and Maly [15].

In Participatory Budgeting (in short PB) our input is a tuple $I = (\mathcal{P}, c, b)$ where $\mathcal{P} = \{p_1, \dots, p_m\}$ is the set of all projects, $c : \mathcal{P} \to \mathbb{R}_{>0}$ is the cost function and $b \in \mathbb{R}_{>0}$ the budget constraint. Given a subset $P \subseteq \mathcal{P}$, we denote by $c(P) = \sum_{p \in P} c(p)$ the cost of P. In PB we want to select an allocation $\pi \subseteq \mathcal{P}$ having $c(\pi) \leq b$. The set of all feasible allocations given an instance I is denoted $F_{eas}(I) = \{\pi \subseteq \mathcal{P} \mid c(\pi) \leq b\}$.

We let $\mathcal{N} = \{1, \dots, n\}$ denote the set of voters who have approval preferences over the set of projects. An approval ballot A_i for a voter $i \in \mathcal{N}$ is the subset of projects that i approves of. Given a project $p \in \mathcal{P}, \mathcal{N}(p)$ is the set of voters who approves of p, $\mathcal{N}(p) = \{i \in \mathcal{N}, p \in A_i\}$. $\mathcal{A} = (A_i)_{i \in \mathcal{N}}$ is the vector of all voters' approval ballots.

A PB rule R takes as an input a tuple $(I, \mathcal{A})$ and gives a set of feasible allocations $R(I, \mathcal{A}) \subseteq F_{eas}(I)$. We usually are more interested in giving a single winning allocation, but we allow for ties.

In order to interpret the approval ballots of the voters, Brill et al. [2] introduced the concept of a satisfaction function and the following definitions.

Definition 1. *(Approval-Based Satisfaction Functions)*

Given an instance $I = (\mathcal{P}, c, b)$ and ballots $\mathcal{A}$, an (approval-based) satisfaction function *is a mapping $sat : 2^{\mathcal{P}} \to \mathbb{R}_{\geq 0}$ satisfying the following two conditions:*

- *$sat(P) \geq sat(P')$ for all $P, P' \subseteq \mathcal{P}$ such that $P \supseteq P'$: the satisfaction is inclusion-monotonic;*
- *$sat(P) = 0$ if and only if $P = \emptyset$: the satisfaction is zero only for the empty set.*

The satisfaction of agent $i \in \mathcal{N}$ for a budget allocation $\pi \in F_{eas}(I)$ is defined as:

$$sat_i(\pi) = sat(\{p \in \pi \mid A_i(p) = 1\}).$$

A satisfaction function is strictly cost-responsive if for $p, p' \in \mathcal{P}, c(p) < c(p') \Rightarrow sat(p) < sat(p')$.

Important satisfaction functions include $sat^{card} : P \mapsto |P|$ that is additive and approval based and $sat^{cost} : P \mapsto c(P)$ that is an additive approval based function.

Now that we have defined what is the voter's satisfaction, we introduce the notion of cohesive group that will be useful in order to define justified representation.

Definition 2. *(cohesive groups)*

Given an instance $I = (\mathcal{P}, c, b)$ of participatory budgeting, and a profile $\mathcal{A}$ of approval ballots, a subset of voters $S \subseteq \mathcal{N}$ is said to be P-cohesive for a set of projects $P \subseteq \mathcal{P}$ if the two following conditions hold:

- $\forall p \in P, i \in S, p \in A_i$
- $\frac{|S|}{n} \times b \geq c(P)$

Definition 3. *(EJR[sat])*

Given an instance $(I, \mathcal{A})$ of participatory budgeting, and a satisfaction function sat, a budget allocation $\pi \in F_{eas}$ is said to satisfy Extended Justified Representation for sat if $\forall P \subseteq \mathcal{P}$ and for all P-cohesive group of voters $S, \exists i \in S$ s.t. $sat_i(\pi) \geq sat_i(P)$.

Following Peters et al. [13] who proved that EJR[sat^{card}] is NP-hard to satisfy, Brill et al. [2] proved that EJR[μ] is also NP-hard to satisfy for any strictly cost responsive satisfaction function μ.

Definition 4. *(EJR[sat]-1+) Given an instance $(I, \mathcal{A})$ and a satisfaction function μ, an outcome $\pi \subseteq \mathcal{P}$ satisfies EJR[μ]-1^+ if for any T-cohesive group of voters $S \subseteq \mathcal{N}$ either $T \subseteq \pi$ or $\exists i \in S$ and $p \in T \setminus \pi$ s.t. $\mu_i(\pi \cup \{p\}) > \mu_i(T)$.*

We also present the fair share axioms introduced in PB by Maly et al. [11].

Definition 5 (Share, Fair Share)**.** *Given an instance of PB $(I, \mathcal{A})$ the share of a voter i for a set of projects $P \subseteq \mathcal{P}$ is share$_i(P) = \sum_{p \in P \cap A_i} \frac{c(P)}{|\mathcal{N}(p)|}$. Its fair share is $FS(i) = \min(b/n, share_i(A_i))$.*

Ideally, and in order to satisfy fair share (FS) one would want to select a set of projects $P \subseteq \mathcal{P}$ such that for every voter $i \in \mathcal{N}$, share$_i(P) \geq fs(i)$, but Maly et al. [11] proved that it is often impossible. In order to make use of this fair share notion, they introduced the next two axioms we will present as well as two measures of distance to fair share we will come back to in the experiments.

Definition 6 (Local Fair Share)**.** *Given a PB instance $(I, \mathcal{A})$ an allocation $P \subseteq \mathcal{P}$ satisfies local FS if there is no project $p \notin P$ such that :*

$$\forall i \in \mathcal{N}(p), share_i(P \cup \{p\}) < FS(i).$$

This definition is close to priceability, having that any contribution function has to spread the projects costs equally bete=ween its supporters. The last definition from Maly et al. [11] we are interested about is Extended Justified Share (EJS) which can be described as EJR[share]+ where the share of a voter is interpreted as a satisfaction function.

Definition 7. *(EJS) Given a PB instance $(I, \mathcal{A})$, an allocation $\pi \subseteq \mathcal{P}$ satisfies EJS (or EJR[share]) if for any $P \subseteq \mathcal{P}$ and any P-cohesive group of voters $S \subseteq \mathcal{N}$ there exists $i \in S$ s.t.*

$$share_i(\pi) \geq share_i(P).$$

Alike for EJR[cost], selecting an outcome satisfying EJS is NP-hard as proved by Maly et al. [11], which is why they introduced EJS-1 that is exactly EJR[share]-1+. They show that MES[share] satisfies both; local FS and EJS while MES[cost] fails to guarantee them.

3 Dynamic MES

3.1 Formal definition

The Method of Equal Shares (MES) is a rule introduced by Peters et al. [13] that satisfies interesting properties and is put forward for real world participatory budgeting [3]. We introduce an adaptation of this rule that we call Dynamic MES.

Definition 8. *Dynamic MES[μ]*

Given a PB instance $(I, \mathcal{A})$ dynamic MES first gives every voter a share of b/n of the budget, and adds projects to a winning set W sequentially as follows.

At each step, given a set of winning projects W, for all $p \in \mathcal{P} \setminus W$, we compute the dynamic affordability of $W \cup \{p\}$:

$$\begin{array}{lll} min & q & \\ s.t. & \forall i \in \mathcal{N}, p' \in W \cup \{p\}, \gamma_i(p') & \leq q \times \mu_i(\{p'\}) \\ & \forall i \in \mathcal{N}, p' \notin A_i \cup (W \cap \{p\}), \gamma_i(p') & = 0 \\ & \forall i \in \mathcal{N}, \sum_{p' \in (W \cup \{p\}) \cap A_i} \gamma_i(p') & \leq b/n \\ & \forall p' \in W \cup \{p\}, \sum_{i \mathcal{N}(p')} \gamma_i(p) & = c(p') \\ & \forall i \in \mathcal{N}, p \in \mathcal{P}, \gamma_i(p) \in [0, b], q \in [0, b] & \end{array}$$

[3] https://equalshares.net

We add the project with minimal dynamic affordability for $W \cup \{p\}$ to the set of winning projects and repeat this step until there is no project $p \in \mathcal{P} \setminus W$ such that $W \cup \{p\}$ is affordable.

The difference with MES is that we allow for a re-allocation of the cost of previously selected projects. In order to compute this dynamic affordability, we can use any linear programming solver, we will provide more details about the computation in the experimental section.

3.2 The Helenka paradox

Papasotiropoulos et al. [12] introduced MES with bounded overspending, we adapt their work in order to define Dynamic MES with bounded overspending. The main advantages of doing so is to have a method that is more flexible as it avoids the Helenka paradox that we re illustrate in table 1, and that is resolute as opposed to classic versions of MES and Dynamic MES.

The Helenka paradox comes from a real world PB instance in Helenka Poland, it was discovered and analyzed by Papasotiropoulos et al. [12]. This is a very critical example as in the first phase of the PB implementation, the citizen of this city made a strategic choice to group projects together in order to make one. This strategic behavior can seem surprising, even more with the MES rule, but it is actually optimal when the PB rule is the standard utilitarian method that is used in most cities. In this example, the total budget is 310k, so only one of the two proposed projects can be selected. The second group of 11 voters is cohesive with respect to B, as it deserves slightly more than 8k according to EJR. Then, in order to satisfy EJR on this instance, any rule has to select project B and project B only as project A is then too expensive. One can easily see that is is not suitable in this example as it would leave 97% of the voters empty handed.

Project	Cost	403 voters	11 voters
A	310k	1	0
B	6k	0	1

Table 1. The Helenka paradox, Papasotiropoulos et al. [12]

There are several ways of avoiding such critical examples, this as to do with the first phase of the digital PB process when citizen and associations propose their projects. One idea is to define a strict policy on merging projects and refuse projects that clearly consist in several independent sub projects. This could be hard to actually implement and to argue as refusing projects or strategic behavior can be seen as undemocratic. Another idea explored in Rey et al. [16](chapter 8) is to build a PB process that is only partly digital. As explained in Wampler et al. [21], the mathematical model we study is mostly used in Europe and North America. Other implementations call for the onsite participation of citizen, open discussions and deliberations in order to reach a set of projects to be funded that can be approved by most. This type of participation has gained a lot of popularity in both real world examples and in research. Such groups can be selected by sortition as explored by Ebadian et al. [6] and Van Reybrouck [20] or by more sophisticated algorithms as Do et al. [5] and Lang and Skowron [10] did. In order to have a more democratic pre selection, one could ask this group to adjust and accept(or reject) the proposed projects and their cost before a more global vote is done.

3.3 Dynamic MES with bounded overspending

We adapt Papasotiropoulos et al.'s [2024] bounded overspending (BOS) algorithm, with the same idea of computing the dynamic affordability of projects at each step.

Definition 9 (Dynamic MES with bounded overspending (DBOS))**.** *DBOS selects projects iteratively until no project can be added to the allocation without overspending the budget b. Given a set $P \subseteq \mathcal{P}$ of previously selected projects, DBOS selects the project minimizing the following optimization problem:*

$$
\begin{array}{lrcl}
\textit{min} & {}^{q}/_{\alpha} & & \\
\textit{s.t.} & \forall i \in \mathcal{N}, p' \in W \cup \{p\}, \gamma_i(p') & \leq & q \times \mu_i(\{p'\}) \\
& \forall i \in \mathcal{N}, p' \notin A_i \cup (W \cap \{p\}), \gamma_i(p') & = & 0 \\
& \forall i \in \mathcal{N}, \sum_{p' \in (W \cup \{p\}) \cap A_i} \gamma_i(p') & \leq & {}^{b}/_{n\alpha} \\
& \forall p' \in W \cup \{p\}, \sum_{i \mathcal{N}(p')} \gamma_i(p) & = & c(p') \\
& \alpha \leq 1 & &
\end{array}
$$

A major issue for the implementation of DBOS is that this optimization problem is non linear but also not even convex. Thus, it is not supported by classic optimization solvers like gurobi or cplex. In order to compute the dynamic affordability of a set of projects, we cannot use the technique Papasotiropoulos et al. [12] used as it relies on the fact that only a fixed number of values are interesting for α, γ because we cannot change how much voters paid for previously selected projects.

Technically, gurobi for python can handle non linear constraints as in our problem, but not non-convex objective functions. Thus we cannot solve directly this, but check if a given value of ${}^{\gamma}/_{\alpha}$ is feasible (we describe our next algorithm supposing we implemented a function feasible(γ) which states if that model is feasible for a given γ). We propose a dichotomic search algorithm that solves our optimization problem up to $\epsilon > 0$.

First observation is that given a set of previously selected projects $P \subseteq \mathcal{P}$, a new project $p \notin P$ and the affordability $\gamma = {}^{\gamma}/_{\alpha}$ of P (that we should have previously computed) the new affordability $\gamma' = {}^{\gamma'}/_{\alpha'}$ must satisfy $\gamma' \geq \min(\gamma, \frac{c(p)}{|\mathcal{N}(p)|})$. Moreover, as it is always possible to add $\frac{c(p)}{|\mathcal{N}(p)|}$ to each voter's budget and make p's supporters evenly pay for adding it without changing the previous cost allocation function, we get: $q' \leq \max(q, \frac{c(p)}{|\mathcal{N}(p)|})$ and $\alpha' \geq \frac{b|\mathcal{N}(p)|\alpha}{b|\mathcal{N}(p)|+n\alpha c(p)}$.

In theory, this algorithm is enough to run the DBOS rule for a given PB instance. In practice though, it is not efficient enough in order to handle real life instance from pabulib, which is why we did not include it in our experiments. Finding an efficient implementation for DBOS will be interesting as Dynamic MES does not support well completion methods as we will explain later.

3.4 Properties of dynamic MES

What is interesting about dynamic MES is that it satisfies most of the proportionality properties MES satisfies.

Theorem 1. *Let μ be an additive satisfaction function, then Dyn MES[μ] satisfies EJR[μ]-1+.*

We adapt the proof of theorem A.1 from Brill et al. [2].

Proof. Let $(I, \mathcal{A})$ be an instance of approval based PB, and let $W = \{p_1, \ldots, p_k\}$ be the outcome output by Dyn MES[μ] on this instance where p_1 is the first selected project, p_2 the second and p_k the last.

Algorithm 1 Dynamic affordability given P

$\alpha, q = 1, 0$
$\gamma = {}^{q}/_{\alpha}$
$P \leftarrow \emptyset$
while $\exists p \notin P, c(p) + c(P) \leq b$ **do**
  $\gamma^\star = +\infty$
  for $p \notin P, c(p) + c(P) \leq b$ **do**
    $\gamma_{\text{inf}} = \min(\gamma, \frac{c(p)}{|\mathcal{N}(p)|})$
    $q_{\text{supp}} = \max(q, \frac{c(p)}{|\mathcal{N}(p)|})$
    $\alpha_{\text{inf}} = \frac{b|\mathcal{N}(p)|\alpha}{b|\mathcal{N}(p)|+n\alpha c(p)}$
    $\gamma_{\text{supp}} = {}^{q_{\text{supp}}}/_{\alpha_{\text{inf}}}$
    while $\gamma_{\text{supp}} - \gamma_{\text{inf}} > \epsilon$ **do**
      $\gamma' = \frac{\gamma_{\text{inf}}+\gamma_{\text{supp}}}{2}$
      if feasible(γ') **then**
        $\gamma_{\text{supp}} = \gamma'$
      else
        $\gamma_{\text{inf}} = \gamma'$
      end if
      if $\gamma_{\text{inf}} > \gamma^\star$ **then**
        Break
      end if
    end while
    if $\gamma_{\text{supp}} < \gamma^\star$ **then**
      $\gamma^\star \leftarrow \gamma_{\text{supp}}$
      $p^\star \leftarrow p$
    end if
  end for
end while
Output $\gamma^\star, p^\star$

Let $S \subseteq \mathcal{N}$ be a T-cohesive group of voters for $T \subseteq \mathcal{P}$. In order to show that Dyn MES[μ] satisfies EJR[μ]-1+, we need to show that either $T \subseteq W$ or $\exists i \in S, p \in W \subseteq T$ s.t. $\mu_i(W \cup \{p\} > \mu_i(T)$. We assume $T \subsetneq W$.

Let $p^\star = \arg\max\{c(p), p \in T \setminus W\}$ be the most expensive project that is not selected from T. Let $p_{k^\star}$ be the first project s.t. after it is selected the load of one voter $i^\star$ in S exceeds $\frac{b}{n} - \frac{c(p^\star)}{|S|}$. The existence of such a project is guaranteed as if it did not exist, the voters from S would have enough money at the end of the algorithm to pay for $p^\star$, and this project would have been selected.

We want to show that $\mu_i(W \cup \{p\}) > \mu_i(T)$, as μ is additive, we have

$$\begin{aligned} & \mu_{i^\star}(W^\star) > \mu_{i^\star}(T \setminus \{p^\star\}) \qquad (1)\\ \Leftrightarrow\ & \mu_{i^\star}(W^\star \cup T) + \mu_{i^\star}(W^\star \setminus T) > \\ & \qquad \mu_{i^\star}(T \setminus (W^\star \cup \{p^\star\})) + \mu_{i^\star}(T \cap W^\star) \\ \Leftrightarrow\ & \mu_{i^\star}(W^\star \setminus T) > \mu_{i^\star}(T \setminus (W^\star \cup \{p^\star\})) \end{aligned}$$

We first work on the right side of the inequality, let α be the affordability of $W^\star$, we have by definition of Dyn MES:

$$\sum_{p \in W^\star \setminus T} \gamma_{i^\star}(p) \leq \alpha \sum_{p \in W^\star \setminus T} \mu_{i^\star}(\{p\}) \Rightarrow$$
$$\mu_{i^\star}(W^\star \setminus T) \geq \frac{1}{\alpha} \sum_{p \in W^\star \setminus T} \gamma_{i^\star}(p).$$

We now work on the left side of the equation, for $p \in T$, we let $q^\star(p) = |S| \times \frac{\mu(p)}{c(p)}$. By letting $q_{\max} = \max q^\star(p), p \in T$, we obtain:

$$\begin{aligned} \mu_{i^\star}(T \setminus (W^\star \cup \{p^\star\})) &= \sum_{p \in T \setminus (W^\star \cup \{p^\star\})} \mu_{i^\star}(p) \\ &= \sum_{p \in T \setminus (W^\star \cup \{p^\star\})} q^\star(p) \frac{c(p)}{|S|} \\ &\leq q_{\max} \frac{c(T \setminus (W^\star \cup \{p^\star\}))}{|S|}. \end{aligned}$$

We now need to prove that

$$\frac{1}{\alpha} \sum_{p \in W^\star \setminus T} \gamma_{i^\star}(p) \geq q_{\max} \times \frac{c(T \setminus (W^\star \cup \{p^\star\}))}{|S|}.$$

We first prove that $\frac{1}{\alpha} \geq q_{\max}$.

Let $p' \in T \setminus (W^\star \cup \{p^\star\})$, by definition of $k^\star$, p' and because S is T-cohesive, p' was $\frac{1}{q^\star(p')}$ affordable from step 1 to step $k^\star$. Then, we can prove by induction on k that given α_k the affordability of $\{p_1, \ldots, p_k\}$, we have $\alpha_k \leq \frac{1}{q(p')}$. Indeed, as voters in S can fully pay for p', the affordability of $\{p_1, \ldots, p_k, p'\}$ cannot exceed $\max(\alpha_k, \frac{1}{q(p')})$.

Thus, we have $\alpha \leq \frac{1}{q(p')} \Rightarrow \frac{1}{\alpha} \geq q_{\max}$.

Eventually, we want to prove that $\sum_{p \in W^\star \setminus T} \gamma_{i^\star}(p) > \frac{|T \setminus (W^\star \cup \{p^\star\})|}{|S|}$.

By definition of $k^\star$, and because S is T-cohesive, we have:

$$l_{i^\star}(W^\star) + \frac{c(p^\star}{|S|} > \frac{b}{n} \geq \frac{c(T)}{|S|} = \frac{c(T \setminus \{p^\star\})}{|S|} + \frac{c(p^\star)}{|S|}$$

.

Then, by definition of the load, we obtain:

$$\sum_{p \in W^\star} \gamma_{i^\star}(p) = l_{i^\star}(W^\star) > \frac{c(T \setminus \{p^\star\}}{|S|})$$

This is equivalent to

$$\begin{aligned} \sum_{p \in T \cap W^\star} \gamma_{i^\star}(p) + & \sum_{p \in W^\star \setminus T} \gamma_{i^\star}(W^\star \setminus T) \\ & > \frac{c(T \setminus (W^\star \cup \{p^\star\})}{|S|}) + \frac{c(T \cap W^\star)}{|S|}. \end{aligned}$$

As no voter in $|S|$ has spent all its budget, they have all paid in full for projects in $W^\star \cap T$, then $\forall p \in W^\star \cap T, \gamma_{i^\star}(p) \leq \frac{c(p)}{|S|}$. Thus, $\sum_{p \in W^\star \cap T} \gamma_{i^\star}(p) \leq \frac{c(T \cap W^\star)}{|S|}$. Therefore, we must have

$$\sum_{p \in W^\star \setminus T} \gamma_{i^\star}(p) > \frac{c(T \setminus (W^\star \cup \{p^\star\}))}{|S|}.$$

Then we obtain 1, so Dyn MES[μ] satisfies EJR[μ]-1+.

□

Theorem 2. *DMES[share] satisfies EJS-1 and local FS.*

Proof. The first result follows directly from theorem 1 as share is an additive satisfaction function.

For local FS, we can simply adapt the proof from Maly et al. [11]. The way DMES works is very similar to that of MES, then as for MES[share], no set of projects can be q-affordable for any $q < 1$

at any step, and there is a step k after which no set of projects is 1-affordable. Let us consider the allocation π_k built by DMES[share] at step k. Then, as π_k was 1-affordable, for any voter $i \in \mathcal{N}$, its total load for $\pi_k : l_i(\pi_k)$ is exactly $l_i(\pi_k) = \text{share}_i(\pi_k)$.

As no other project is 1-affordable, we get that $\forall p \notin \pi_k, \exists i \in \mathcal{N}(p)$ s.t. $b/n - l_i(\pi_k) < share_i(\pi_k)$ then as $l_i(\pi_k) = \text{share}_(\pi_k)$, we get $\text{share}_i(\pi_k \cup \{p\}) = \text{share}_i(\pi_k) + \text{share}_i(\{p\}) > b/n \geq FS(i)$.

Therefore, DMES[share] satisfies local FS.

□

3.5 Key differences between MES variations

Maly et al. [11] designed a MES variation (MES[share]) in order to obtain fairer outcomes than MES[cost] (by being closer to satisfy fair share). Unfortunately, their experiments showed that MES[cost] was outperforming MES[share] on their measures (distance to fair share and fair share ratio). We will give some examples of instances in order to give an intuition for what we think are the reasons for MES[share] failing to beat MES[cost] on fair share measures, and why we thought DMES[cost] and DMES[share] would be interesting. Table 2 gives an example with 4 voters and 5 projects where MES[cost] is closer to satisfy FS than MES[share].

First, we remind the reader of the fair share measures introduced by Maly et al. [11]. The average distance to Fair Share (dist_{FS}) is defined as

$$\text{dist}_{FS}(W) = \frac{1}{n} \sum_{i \in \mathcal{N}} |\text{share}(A_i \cap W) - FS(i)|$$

They also define the average capped fair share ratio (FS_{ratio}) as

$$FS_{\text{ratio}} = \sum_{i \in \mathcal{N}} \min(1, \frac{share(A_i \cap W)}{FS(i)})$$

The key difference between the two measures is that in order to minimize the distance to fair share, not only do we need to try and give every voter at least their fair share, but we also need not to give them too much (more than their fair share) which does not always make sense. On the other hand, the capped fair share ratio only measures how far every voter is from getting their fair share which is more relevant in our opinion. Consider an example with a budget of 6, three voters and two projects: project 1 costs 2 units and Project 2 costs 1. Both voters 1 and 2 approve on project 1 while only voter 2 approves of project 2. In order to minimize the distance to fair share, we need to select only project 1 as otherwise, voter 2 would get strictly more than its fair share which we think is not relevant. Another important note on the fair share axioms and measures is that it does not seem to us like a good objective to optimize but remains an interesting measure of proportionality and fairness. Indeed, for any of the two measures, if 51% of the voters approve on one single project that costs 100% of the budget, one has to select this one project in order to optimize these measures regardless of the other voters preferences. Then, using a rule like a fair share maximizer as in Maly et al. [11] is not suitable for real world application (which they already stated for computation reasons). Nevertheless, we think these measures remain interesting for measuring proportionality in PB as focusing only on EJR has limitations because cohesive groups are rare in real world instances as noted by Faliszewski et al. [7].

If we compute the average distance to fair share MES[cost] has $1/16$ and MES[share] $1/4$, which makes MES[cost] more interesting on this outcome. Interestingly, while both rules are resolute on this example, MES[cost] uses all the budget while MES[share] uses only 6 out of 8. In our opinion, the main reason for this is that MES[share] tends to select projects with low cost first so the projects that make it fail FS are more critical. It is very clear on this example.

	1	2	3	4	cost	MES[cost]	MES[share]
p_1	✓	✓	✓	✗	3	1	1
p_2	✗	✗	✓	✓	3	1	0
p_3	✓	✗	✗	✗	0.1	10	10
p_4	✗	✓	✗	✗	0.1	10	10
p_5	✗	✗	✗	✓	0.1	0	10

Table 2. Example with a budget of 8, p_3, p_4 and p_5 appear 10 times

An interesting feature of DMES compared to MES is also related to what we just showed: as it is allowed to change previous voters' contributions, it can get closer to using full budget than MES, even with completion as we can see in the example from table 3.

	1	2	3	4	cost	MES[cost]	DMES[cost]
p_1	✓	✓	✓	✗	5	1	1
p_2	✓	✓	✗	✗	3	0	1
p_3	✓	✗	✗	✗	0.3	1	0

Table 3. Example with a budget of 8, advantage of DMES over MES

In this example, MES[cost] is resolute as it selects $\{p_1, p_3\}$ and p_2 cannot be afforded, but DMES[cost] is using the full budget while MES[cost] is using only 5.3 out of 8. If we look at the distance to fair share, MES[cost] performs better than DMES[cost] on this instance, having an average distance of 0.22 against 0.55. As we mentioned before, distance to fair share tends to penalize outcomes where some voters obtain more than their share, which is why we prefer to focus on fair share ratio. DMES[cost] has a fair share ratio of 0.94 on this instance whil MES[cost] has 0.89. This example is close to the Helenka paradox; one could argue that MES[cost] selected a better outcome as it satisfies EJR[cost] while DMES[cost] fails it, but selects an outcome that satisfies more voters.

Important to note is that DMES[cost] is more efficient than fair, as example 4 shows. In this example there is an outcome (selecting all but project 2) that is clearly more fair than others and is not that of DMES[cost].

	1	2	3	4	cost	DMES[cost]	DMES[share] and MES[cost]
p_1	✓	✓	✓	✗	5	✓	✓
p_2	✗	✗	✓	✓	3	✓	✗
p_3	✓	✗	✗	✗	1/3	✗	✓
p_4	✗	✓	✗	✗	1/3	✗	✓
p_5	✗	✗	✗	✓	2	✗	✓

Table 4. Example with a budget of 8, advantage of DMES[share] on DMES[cost]

These intuitions lead us to think DMES[share] could be a good alternative to MES[cost], but this needs to be checked using experiments based on real world data.

4 Experiments

We ran experiments on real world data from pabulib, Faliszewski et al. [7]. Our implementation of Dynamic MES uses an LP solver (Gurobi) where the number of constraints is linear to the number of

projects times the number of different approval profiles. Having this large number of variables and constraints led Gurobi to fail computing DMES on a large number of real world instances. This limited us to use 217 of the real world approval instances out of the 745 available on pabulib for the methods without completion and only 34 with completion. This is a major issue for DMES which is not fit for real world instances as it is now, finding a more efficient implementation of our rule would be interesting if we want to tackle larger size instances.

We compared our Dynamic MES algorithm with the classic MES with and without completion, measuring 5 indices: The average satisfaction of voters is the average cost of the selected projects that a voter approves on, the Camberlin-Courant (CC) score is the percentage of voters that approved at least one selected project, and the Gini index measures the inequality of satisfaction among the voters (the lower the better). These three indexes were in the original paper by Faliszewski et al. [7], while Maly et al. [11] introduced to other fair share related indexes that we also used in our experiments. The distance to fair share measures how far an allocation is to satisfy the fair share axiom introduced by Maly et al. [11] in absolute difference while the fair share ratio gives the ratio distance with fair share (it sums the ratios of share over FS on each voters). We think the fair share ratio is more interesting to study as a voter receiving more than its fair share does not give penalties, we can then focus on voters who received less than their fair share.

4.1 Comparing rules without completion

Our first experiment consisted in comparing MES[cost] with DMES[cost] and DMES[share] on our 5 measures without completion on any of the rules. We did this for two reasons: first, we think it is important to see if a rule is more efficient in itself of because of the completion, second reason for that is the running time of DMES that is a lot longer than that of MES, so this was an easy first step of comparison. Figure 1 shows a radar comparison of the three rules on our 5 measures (using average and madian scores of the rules).

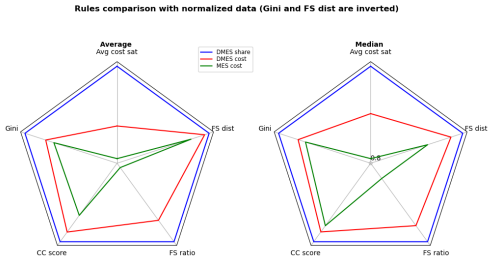


Figure 1. Radar comparison of MES variations, Gini and FS ratio are inversed

Figure 1 shows a clear order on the performances of the three rules without completion: DMES[share] outperforms DMES[cost] which beats MES[cost]. It is not surprising that DMES[share] beats DMES[cost] because of the observations we made in subsection 3.5. Both of them beating MES is also logical as DMES tends to select more projects than MES as it is more flexible. A good surprise nonetheless is the fact that both DMES versions beat MES[cost] on Gini index as this measure is not dependent of the amount of budget spent like others are.

An important thing to note is that the differences between the three rules are only slight. DMES[share] has an average voter satisfaction of 103 838 while DMES[cost] has 96 957 and MES[cost] 93 179. On Gini index, the order is less clear as we get the same order regarding average and median score ([0.348, 0.357, 0.369],[0.316, 0.330, 0.336]) but MES[cost] performs better on the worst case than DMES[cost]. We put the average measures of each rules in table 5.

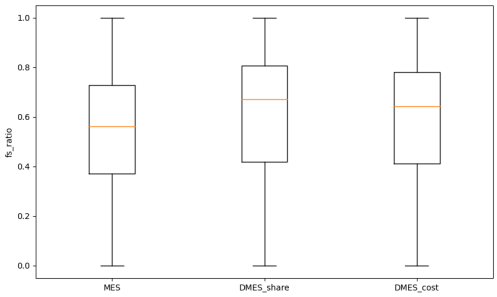


Figure 2. Boxplot comparison of rules without completion on FS ratio

What we are more interested about is the distance to fair share and fair share ratio. While the difference is clear regarding fair share ratio with DMES[share] being 9% better than MES[cost], the difference is only 1.8% on distance to fair share. It was to be expected as we discussed in subsection 3.5, we mostly aimed at improving fair share ratio by using DMES[share].

Measure	MES[cost]	DMES[cost]	DMES[share]
Avg sat	93 179	96 957	103 868
CC score	0.736	0.753	0.763
Gini	0.360	0.357	0.348
Dist FS	5043	4962	4936
FS ratio	0.551	0.596	0.614

Table 5. Average score of the three rules without completion

In conclusion, the performances of DMES[share] and DMES[cost] are better than those of MES[share] on the real world instance we tested. This is not enough to make DMES being better than MES for several reasons. First, as we explained before, our implementation of DMES is far from being as efficient as MES implementation and we did not test it on all pabulib data. Moreover, the state of the art algorithm (SOTA) in PB is MES with completion so we need to compare our rules with completion in order to draw a final conclusion. Indeed, as illustrated in figure 3, when we compare the three rules we study to Phragmen which is resolute but not the SOTA for PB, it beats them on all indices but Gini. This is promising as Gini is the only index among the 5 that can be high without having a resolute outcome. Thus, one might expect that having DMES beating MES and Phragmen without completion means that this rule is more fair in that the satisfaction of voters are well distributed.

4.2 Comparing rules with completion

The second experiment consisted in comparing our three rules when using the add1 completion method proposed by Peters et al. [13]. Given a PB rule $\mathcal{F}$, the completion with add1 consists in running this rule $\mathcal{F}$, if it is not exhaustive, we increase the budget by the number of voters and re run $\mathcal{F}$ and repeat until having an exhaustive outcome. As noted by Kraiczy and Elkind [8], this completion method is computationally heavy even for the original MES rule, which makes it

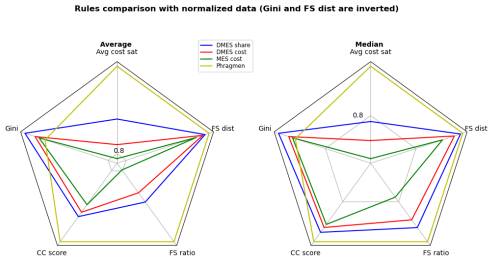


Figure 3. Radar comparison, adding Phragmen that is resolute

even worse for DMES as we have to run our algorithm repeatedly. As we mentioned before, we were only able to run our DMES algorithm with the add1 completion rule because of high computational cost.

Measure	MES[cost]	DMES[cost]	DMES[share]
Avg sat	121 799	113 031	122 043
CC score	0.904	0.898	0.906
Gini	0.289	0.298	0.288
Dist FS	2348	2376	2306
FS ratio	0.800	0.781	0.802

Table 6. Average score of the three rules with completion

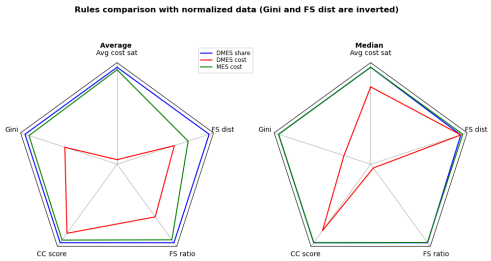


Figure 4. Radar comparison of the three rules with add1 completion

As one can see with table 6 and figure 4, the results with completion are not as positive for DMES as without completion. First, as we expected (see section 3.5), DMES[cost] is a lot less interesting when we use a completion rule. While DMES[share] still perfoms better than MES[cost] with the completion method, the difference is very tight, with only 0.1% for average cost, 2% for CC score, 3% for Gini index, 2% for distance to FS and 2.5% for FS ratio.

Having only this small of an improvement compared to the SOTA algorithm on the measures with an explosion of the computational cost comforts MES[cost] as the best known algorithm for PB to date.

5 Conclusion

Building on the work of Sánchez-Fernández et al. [17] and Maly et al. [11], we have introduced a new variation of the method of equal shares with the aim of getting fairer outcomes. Our rule; dynamic MES keeps the axiomatic guarantees that MES satisfies while showing better performance on real world pabulib data especially when we do not consider completion methods. The key difference between MES and dynamic MES is that MES computes the affordability of upcoming projects having the contribution of voters for previously selected projects fixed while dynamic MES computes the affordability of the set of selected projects with the potential new project to be added. While this approach has a good motivation (exploring MES variations to better understand this rule) and shows good perfomrance on real world PB data from pabulib, a major limitation of this rule is its computation.

Computing the dynamic affordability of adding a new projects is very demanding and we did not find an efficient algorithm to do so, having to solve an LP via Gurobi. Moreover, using its bounded overspending version as in Papasotiropoulos et al. [12] does not seem reasonable for now as the optimization problem is no longer linear and not even concave. Finding efficient algorithms for computing these are interesting directions for future work, as our experiment show that DMES[share] has better performances on real world PB data when it is computable through Gurobi. One could also explore more efficient completion methods like Kraiczy and Elkind [8] as we were able to run our algorithm without completion on approximately one third of the instance but our algorithm with completion on only on 5% on the instances from pabulib.

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